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# ON ALMOST BLOW-ANALYTIC EQUIVALENCE

GOULWEN FICHOU AND MASAHIRO SHIOTA

**ABSTRACT.** We study the analytic equivalence of real analytic function germs after desingularization and state the cardinality of the classes under this equivalence relation. We consider also the Nash case, and compare these equivalences with the blow-analytic (respectively blow-Nash) equivalence. We prove an approximation result after desingularization: Nash function germs that are analytically equivalent after analytic desingularizations are Nash equivalent after Nash desingularizations.

In the study of real analytic function singularities, the choice of a relevant equivalence relation is a crucial but difficult topic. After Hironaka Desingularization Theorem [12], which enables to produce functions with only normal crossing singularities after a finite sequence of blowings-up along smooth analytic centers, it seems natural to expect that equivalent real analytic functions should admit similar resolutions of their singularities. In that spirit, we propose in this paper to study the equivalence relation obtained by requiring that two real analytic functions, defined on a compact real analytic manifold, are equivalent if there exist two Hironaka desingularizations such that the modified functions with only normal crossing singularities, obtained after the desingularizations, become analytically equivalent.

This definition is weaker than that of blow-analytic equivalence introduced by T. C. Kuo [14], which seems to be, up to now, the best candidate to be the real counterpart of the topological equivalence for complex analytic functions (cf [10] for a recent survey). Let  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$  be analytic function germs. They are blow-analytically equivalent in the sense of [14] if there exist real modifications  $\beta_f : M_f \rightarrow \mathbb{R}^n$  and  $\beta_g : M_g \rightarrow \mathbb{R}^n$  and an analytic isomorphism  $\Phi : (M_f, \beta_f^{-1}(0)) \rightarrow (M_g, \beta_g^{-1}(0))$  which induces a homeomorphism  $\phi : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  such that  $f = g \circ \phi$ . In particular, the desingularizations are replaced by the notion of real modifications (in order to obtain an equivalence relation) and the analytic isomorphism between the modifications induces a homeomorphism.

The equivalence relation we propose to study in this paper is called almost Blow-analytic equivalence in the language of blow-analytic theory (cf [15] and [9] for the terminology). Almost Blow-analytic equivalence is really a different relation: we exhibit in section 1.2 two functions that are almost Blow-analytically equivalent but not blow-analytically equivalent. Note that we do not know in general if these relations are effectively equivalence relations [9] (in particular we use the generated equivalence relations). However we give a proof in proposition 1.4 of the known fact that this is indeed the case if we allow non smooth blowings-up in the definitions. We focus also on the questions of the cardinality of the equivalence classes. We prove in particular that the cardinality of the set of equivalence classes for almost Blow-analytic equivalence is countable. Therefore it is reasonable to hope for a classification! The proof of that result is based on the study of the cardinality for the analytic equivalence classes of normal crossing functions made in [6], where we reduced the problem to the analogous study for Nash functions, i.e. real analytic functions whose graph is a semi-algebraic set (described by a set of polynomial equalities and inequalities with real coefficient polynomials). Actually, the study of almost Blow-Nash

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equivalence for Nash functions presents a great interest by itself (cf. [13, 5] for notions on blow-Nash equivalence). Notably the cardinality of the set of equivalence classes remains countable even if the underlying Nash manifold is no longer compact.

We prove in theorem 1.11 that almost Blow-analytically equivalent Nash function germs are almost Blow-Nash equivalent. This result can be view as a Nash Approximation Theorem [3] after blowings-up. Note that this question remains open for blow-analytic equivalence.

In order to prove theorem 1.11, we focus on the Nash approximation of a Hironaka desingularization of a Nash function. Namely, let  $f$  be a Nash function on a Nash manifold  $M$  and  $X$  a compact semialgebraic subset of  $M$ . Then  $f$  is in particular a real analytic function on a real analytic manifold, and by Hironaka Desingularization Theorem [12] there exists a composition  $\pi$  of blowings-up along smooth analytic centers such that  $f \circ \pi$  has only normal crossing singularities on a neighborhood of  $\pi^{-1}(X)$ . As a main result of the paper, we prove in theorem 2.4 that each blowing-up along a smooth analytic center can be approximated by a blowing-up along a smooth Nash center in such a way that the normal crossing property of the modified function continues to hold. In order to approximate a sequence of blowings-up along smooth centers, we focus in section 2.2 on a Euclidean realization of such a sequence to describe precisely its behavior under a perturbation of the defining ideal of the centers (cf. lemma 2.2). Combined with Néron Desingularization [21], this implies theorem 2.4. But this is not sufficient to prove theorem 1.11 since we need to approximate also the analytic diffeomorphism of the equivalence after the desingularization. To this aim, we need to generalize the Nash Approximation Theorem in [3] to a more general noncompact situation (cf. proposition 3.1). We obtain as a corollary that analytically equivalent Nash function germs on a compact semialgebraic set in a Nash manifold are Nash equivalent (cf. theorem 3.2). The last section is devoted to the proof of theorem 1.11.

In this paper a manifold means a manifold without boundary, analytic manifolds and maps mean real analytic ones unless otherwise specified, and  $\text{id}$  stands for the identity map.

## 1. ALMOST BLOW-ANALYTIC EQUIVALENCE

**1.1. Definition of almost Blow-analytic equivalence.** The classical right equivalence between analytic function germs says that real analytic function germs  $f, g : M \rightarrow \mathbf{R}$  on an analytic manifold  $M$  are equivalent if there exists an analytic diffeomorphism  $h : M \rightarrow M$  such that  $f = g \circ h$ . We are interested in this paper in weaker notions of equivalence between function germs.

**Definition 1.1.** Let  $M$  be an analytic manifold and  $f, g : M \rightarrow \mathbf{R}$  analytic functions on  $M$ . Then  $f$  and  $g$  are said to be *almost Blow-analytically equivalent* if there exist two compositions of finite sequences of blowings-up along smooth analytic centers  $\pi_f : N \rightarrow M$  and  $\pi_g : L \rightarrow M$  and an analytic diffeomorphism  $h : N \rightarrow L$  so that  $f \circ \pi_f = g \circ \pi_g \circ h$ . In the case where there exist such compositions of finite sequences of blowings-up along smooth analytic centers  $\pi_f : N \rightarrow M$  and  $\pi_g : L \rightarrow M$ , and analytic diffeomorphisms  $h : N \rightarrow L$  and  $\tau : \mathbf{R} \rightarrow \mathbf{R}$  such that  $\tau \circ f \circ \pi_f = g \circ \pi_g \circ h$ ,  $f$  and  $g$  are called *almost Blow-analytically R-L (=right-left) equivalent*.

In this paper, we will be interested also in *almost Blow-analytic (R-L) equivalence* for germs of analytic functions, whose definition is similar to definition 1.1.

Here and from now on, we impose some restrictions to the blowings-up, that are natural thanks to Hironaka Desingularization Theorem. In particular, we treat only the case where the images of the centers of the blowings-up of  $\pi_f$  and  $\pi_g$  are contained in their singular

point sets  $\text{Sing } f$  and  $\text{Sing } g$  respectively, and the center  $C$  of each blowing-up is of co-dimension strictly bigger than one. We assume also that  $C$  has only normal crossing with the union  $D$  of the inverse images of the previous centers, i.e. there exists an analytic local coordinate system  $(x_1, \dots, x_n)$  at each point of  $C$  such that  $C = \{x_1 = \dots = x_k\}$  and  $D = \{x_{i_1} \dots x_{i_l} = 0\}$  for some  $1 < k \in \mathbf{N}$  and  $1 \leq i_1 < \dots < i_l \leq n \in \mathbf{N}$ , where  $\mathbf{N} = \{0, 1, \dots\}$ .

We recall that a *semi-algebraic* set is a subset of a Euclidean space which is described by finitely many equalities and inequalities of polynomial functions. A *Nash manifold* is a  $C^\omega$  submanifold of a Euclidean space which is semi-algebraic. A *Nash function* on a Nash manifold is a  $C^\omega$  function with semi-algebraic graph. A *Nash subset* is the zero set of a Nash function on a Nash manifold. The story of Nash manifolds and Nash maps begins with the fundamental paper [17] of J. Nash who realized any compact smooth manifold as a union of some connected components of a real algebraic set. We refer to [20] for an overview on Nash functions on a Nash subset.

**Definition 1.2.** Let  $M$  be a Nash manifold,  $X \subset M$  a semi-algebraic subset and  $f, g$  Nash function germs on  $X$  in  $M$ . Then  $f$  and  $g$  are said to be *almost Blow-Nash equivalent* if there exist open semi-algebraic neighborhoods  $U$  and  $V$  of  $X$  in  $M$ , two compositions of finite sequences of blowings-up along smooth Nash centers  $\pi_f : N \rightarrow U$  and  $\pi_g : L \rightarrow V$  and a Nash diffeomorphism  $h$  from an open semi-algebraic neighborhood of  $\pi_f^{-1}(X)$  in  $N$  to one of  $\pi_g^{-1}(X)$  in  $L$  so that  $f$  and  $g$  are supposed to be defined on  $U$  and  $V$ , respectively,  $f \circ \pi_f = g \circ \pi_g \circ h$  and  $h(\pi_f^{-1}(X)) = \pi_g^{-1}(X)$ .

We define *almost Blow-Nash R-L equivalence* similarly to definition 1.1.

**Remark 1.3.** We are interested in this paper in global versions of the classical blow-analytic equivalence defined by T.-C. Kuo, which considers only germs of functions at a point. In the analytic case, we will consider the local case and also the compact case since the desingularization theorem of H. Hironaka [12] is valid in these situations. In the Nash category, we can deal with even the non-isolated situation by [12, 1].

The *Blow-analytic equivalence* is defined by requiring above  $h$  to induce a homeomorphism of  $M$ . We do not know whether the almost Blow-analytic (R-L) equivalence and the Blow-analytic (R-L) equivalence give equivalence relations (cf. [9]), though this is the case for the blow-analytic equivalence by T.-C. Kuo [14] (another advantage of the definition of the blow-analytic equivalence by T.-C. Kuo is that for an analytic function  $f$  on an analytic manifold  $M$  there exists a real modification  $\beta : M_f \rightarrow M$  such that  $f \circ \beta$  has only normal crossing singularities, by Hironaka Desingularization Theorem). The problem comes from transitivity. However, if we admit blowings-up along non-smooth analytic centers in the definitions of Blow-analytic equivalence and almost Blow-analytic equivalence, we can face this issue.

**Proposition 1.4.** *Allowing non-smooth analytic centers in the definitions, Blow-analytic equivalence and almost Blow-analytic equivalence become equivalence relations.*

*Proof.* Let  $\mathcal{O}_{M_1}$  denote the sheaf of analytic function germs on an analytic space  $M_1$ . For a morphism  $g : M_1 \rightarrow M_2$  of analytic spaces, as locally ringed spaces, and for a sheaf of  $\mathcal{O}_{M_2}$ -ideals  $\mathcal{I}$ , let  $g^{-1}\mathcal{I} \cdot \mathcal{O}_{M_1}$  denote the inverse image ideal sheaf, i.e., the sheaf of  $\mathcal{O}_{M_1}$ -ideals generated by the image of the inverse image  $g^{-1}\mathcal{I}$  of  $\mathcal{I}$  (see [11]). Let  $f_i$ ,  $i = 1, \dots, 4$ , be analytic functions on an analytic manifold  $M$  with  $f_1$  almost Blow-analytically equivalent to  $f_2$ ,  $f_3$  almost Blow-analytically equivalent to  $f_4$  and  $f_2 = f_3$ . Namely, let

$$N_{i,k_i} \xrightarrow{\pi_{i,k_i}} \dots \longrightarrow N_{i,1} \xrightarrow{\pi_{i,1}} N_{i,0} = M, \quad \text{for } i = 1, \dots, 4$$

be sequences of blowings-up with respect to coherent sheaves of non-zero  $\mathcal{O}_{N_{i,j}}$ -ideals, and let  $\tau_2 : N_{1,k_1} \rightarrow N_{2,k_2}$  and  $\tau_4 : N_{3,k_3} \rightarrow N_{4,k_4}$  be isomorphisms of locally ringed spaces such

that

$$f_i \circ \pi_{i,1} \circ \cdots \circ \pi_{i,k_i} \circ \tau_i = f_{i-1} \circ \pi_{i-1,1} \circ \cdots \circ \pi_{i-1,k_{i-1}},$$

$i = 2, 4$ . If the compositions

$$N_{2,k_2} \xrightarrow{\pi_{2,k_2}} \cdots \longrightarrow M \quad \text{and} \quad N_{3,k_3} \xrightarrow{\pi_{3,k_3}} \cdots \longrightarrow M$$

coincide each other, then  $\tau_4 \circ \tau_2$  is an isomorphism from  $N_{1,k_1}$  to  $N_{4,k_4}$  and

$$f_4 \circ \pi_{4,1} \circ \cdots \circ \pi_{4,k_4} \circ \tau_4 \circ \tau_2 = f_1 \circ \pi_{1,1} \circ \cdots \circ \pi_{1,k_1}.$$

Hence it suffices to reduce the problem to the case where

$$N_{2,k_2} \xrightarrow{\pi_{2,k_2}} \cdots \longrightarrow M \quad \text{and} \quad N_{3,k_3} \xrightarrow{\pi_{3,k_3}} \cdots \longrightarrow M$$

coincide.

We will use the following fact. Let  $\mathcal{J}_1$  and  $\mathcal{J}_2$  be coherent sheaves of non-zero  $\mathcal{O}_{M_1}$ -ideals on a reduced and irreducible analytic space  $M_1$ , and let  $g_i : L_i \rightarrow M_1$  denote the blowing-up with respect to  $\mathcal{J}_i$  for  $i = 1, 2$ . Let  $h_1 : N_1 \rightarrow L_1$  and  $h_2 : N_2 \rightarrow L_2$  denote the blowings-up with respect to  $g_1^{-1}\mathcal{J}_2 \cdot \mathcal{O}_{L_1}$  and  $g_2^{-1}\mathcal{J}_1 \cdot \mathcal{O}_{L_2}$ , respectively. Then there exists an unique isomorphism  $\tau : N_1 \rightarrow N_2$  such that  $g_1 \circ h_1 = g_2 \circ h_2 \circ \tau$ .

$$\begin{array}{ccc} N_1 \cong N_2 & \xrightarrow{h_2} & L_2 \\ h_1 \downarrow & & \downarrow g_2 \\ L_1 & \xrightarrow{g_1} & M_1 \end{array}$$

Actually, by the universal property theorem of the blowing-up (see [11] in the algebraic case) apply to the blowing-up  $g_2 : L_2 \rightarrow M_1$  and the morphism  $g_1 \circ h_1 : N_1 \rightarrow M_1$ , there exists an unique morphism  $\pi : N_1 \rightarrow L_2$  such that  $g_2 \circ \pi = g_1 \circ h_1$  since

$$(g_1 \circ h_1)^{-1}\mathcal{J}_2 \cdot \mathcal{O}_{N_1} (= h_1^{-1}(g_1^{-1}\mathcal{J}_2 \cdot \mathcal{O}_{L_1}) \cdot \mathcal{O}_{N_1})$$

is invertible. Next, considering the blowing-up  $h_2 : N_2 \rightarrow L_2$  and the morphism  $\pi : N_1 \rightarrow L_2$ , we obtain a unique morphism  $\tau : N_1 \rightarrow N_2$  such that  $\pi = h_2 \circ \tau$  since

$$\pi^{-1}(g_2^{-1}\mathcal{J}_1 \cdot \mathcal{O}_{L_2}) \cdot \mathcal{O}_{N_2} (= (g_2 \circ \pi)^{-1}\mathcal{J}_1 \cdot \mathcal{O}_{N_2} = (g_1 \circ h_1)^{-1}\mathcal{J}_1 \cdot \mathcal{O}_{N_2} = h_1^{-1}(g_1^{-1}\mathcal{J}_1 \cdot \mathcal{O}_{L_1}) \mathcal{O}_{N_2})$$

is invertible. Then

$$g_1 \circ h_1 = g_2 \circ \pi = g_2 \circ h_2 \circ \tau.$$

By the same reason, we have an unique morphism  $\tau' : N_2 \rightarrow N_1$  such that  $g_1 \circ h_1 \circ \tau' = g_2 \circ h_2$ . Hence  $\tau$  is an isomorphism.

By this fact, we obtain a commutative diagram of blowings-up with respect to coherent sheaves of  $\mathcal{O}_{M_{i,j}}$ -ideals:

$$\begin{array}{ccccccc} M_{k_2,k_3} & \xrightarrow{\nu_{k_2,k_3}} & M_{k_2-1,k_3} & \longrightarrow & \cdots & \longrightarrow & M_{0,k_3} \\ \mu_{k_2,k_3} \downarrow & & \mu_{k_2-1,k_3} \downarrow & & & & \mu_{0,k_3} \downarrow \\ M_{k_2,k_3-1} & \xrightarrow{\nu_{k_2,k_3-1}} & & \longrightarrow & \cdots & \longrightarrow & M_{0,k_3-1} \\ \downarrow & & \downarrow & & & & \downarrow \\ \vdots & & \vdots & & & & \vdots \\ \downarrow & & \downarrow & & & & \downarrow \\ M_{k_2,0} & \xrightarrow{\nu_{k_2,0}} & M_{k_2-1,0} & \longrightarrow & \cdots & \longrightarrow & M_{0,0} \end{array}$$

such that

$$M_{k_2,0} \xrightarrow{\nu_{k_2,0}} \cdots \xrightarrow{\nu_{1,0}} M_{0,0} \quad \text{and} \quad M_{0,k_3} \xrightarrow{\mu_{0,k_3}} \cdots \xrightarrow{\mu_{0,1}} M_{0,0}$$

coincide with

$$N_{2,k_2} \xrightarrow{\pi_{2,k_2}} \cdots \xrightarrow{\pi_{2,1}} M \quad \text{and} \quad N_{3,k_3} \xrightarrow{\pi_{3,k_3}} \cdots \xrightarrow{\pi_{3,1}} M$$

respectively. In particular,

$$\pi_{3,1} \circ \cdots \circ \pi_{3,k_3} \circ \nu_{1,k_3} \circ \cdots \circ \nu_{k_2,k_3} = \pi_{2,1} \circ \cdots \circ \pi_{2,k_2} \circ \mu_{k_2,1} \circ \cdots \circ \mu_{k_2,k_3}.$$

Since  $\tau_2 : N_{1,k_1} \rightarrow M_{k_2,0}$  is an isomorphism, we obtain a sequence of blowings-up

$$N_{1,k_1+k_3} \xrightarrow{\pi_{1,k_1+k_3}} \cdots \xrightarrow{\pi_{1,k_1+1}} N_{1,k_1}$$

and several isomorphisms

$$\tau_{2,k_3} : N_{1,k_1+k_3} \rightarrow M_{k_2,k_3}, \quad \dots, \quad \tau_{2,0} = \tau_2 : N_{1,k_1} \rightarrow M_{k_2,0}$$

such that the following diagram is commutative.

$$\begin{array}{ccccccc} N_{1,k_1+k_3} & \xrightarrow{\pi_{1,k_1+k_3}} & N_{1,k_1+k_3-1} & \longrightarrow & \cdots & \xrightarrow{\pi_{1,k_1+1}} & N_{1,k_1} \\ \tau_{2,k_3} \downarrow & & \tau_{2,k_3-1} \downarrow & & & & \tau_{2,0} \downarrow \\ M_{k_2,k_3} & \xrightarrow{\mu_{k_2,k_3}} & M_{k_2,k_3-1} & \longrightarrow & \cdots & \xrightarrow{\mu_{k_2,1}} & M_{k_2,0}. \end{array}$$

Then the following equalities hold:

$$\begin{aligned} f_2 \circ \pi_{2,1} \circ \cdots \circ \pi_{2,k_2} \circ \mu_{k_2,1} \circ \cdots \circ \mu_{k_2,k_3} \circ \tau_{2,k_3} &= \cdots = \\ f_2 \circ \pi_{2,1} \circ \cdots \circ \pi_{2,k_2} \circ \tau_{2,0} \circ \pi_{1,k_1+1} \circ \cdots \circ \pi_{1,k_1+k_3} &= \\ f_1 \circ \pi_{1,1} \circ \cdots \circ \pi_{1,k_1} \circ \pi_{1,k_1+1} \circ \cdots \circ \pi_{1,k_1+k_3}. \end{aligned}$$

In the same way we obtain a commutative diagram:

$$\begin{array}{ccccccc} M_{k_2,k_3} & \xrightarrow{\nu_{k_2,k_3}} & M_{k_2-1,k_3} & \longrightarrow & \cdots & \xrightarrow{\nu_{1,k_3}} & M_{0,k_3} \\ \tau_{4,k_2} \downarrow & & \tau_{4,k_2-1} \downarrow & & & & \tau_{4,0} \downarrow \\ N_{4,k_2+k_4} & \xrightarrow{\pi_{4,k_2+k_4}} & N_{4,k_2+k_4-1} & \longrightarrow & \cdots & \xrightarrow{\pi_{4,k_4+1}} & N_{4,k_4} \end{array}$$

where the horizontal morphisms are blowings-up and the vertical ones are isomorphisms, with  $\tau_{4,0} = \tau_4$ , and

$$\begin{aligned} f_4 \circ \pi_{4,1} \circ \cdots \circ \pi_{4,k_4} \circ \pi_{4,k_4+1} \circ \cdots \circ \pi_{4,k_2+k_4} \circ \tau_{4,k_2} &= \\ f_3 \circ \pi_{3,1} \circ \cdots \circ \pi_{3,k_3} \circ \nu_{1,k_3} \circ \cdots \circ \nu_{k_2,k_3}. \end{aligned}$$

Therefore

$$\begin{aligned} f_4 \circ \pi_{4,1} \circ \cdots \circ \pi_{4,k_2+k_4} \circ \tau_{4,k_2} \circ \tau_{2,k_3} &= f_3 \circ \pi_{3,1} \circ \cdots \circ \nu_{k_2,k_3} \circ \tau_{2,k_3} = \\ f_2 \circ \pi_{2,1} \circ \cdots \circ \mu_{k_2,k_3} \circ \tau_{2,k_3} &= f_1 \circ \pi_{1,1} \circ \cdots \circ \pi_{1,k_1+k_3} \end{aligned}$$

which proves the transitivity for almost Blow-analytic equivalence. Considering Blow-analytic equivalence, i.e. if moreover  $\tau_2$  and  $\tau_4$  induce homeomorphisms of  $M$ , then  $\tau_{4,k_2} \circ \tau_{2,k_3}$  induces automatically a homeomorphism of  $M$ .  $\square$

**1.2. Blow-analytic versus almost Blow-analytic equivalence.** In  $\mathbf{R}^2$ , blow-analytic equivalence coincides with almost Blow-analytic equivalence. We prove in this section that these relations do not coincide in general, by giving an explicit example in  $\mathbf{R}^4$  of almost Blow-analytically equivalent functions whose germs at 0 are not blow-analytically equivalent.

Let  $f$  and  $g$  be the  $C^\omega$  functions on  $\mathbf{R}^4$  in variables  $(u, v, w, x)$  defined by

$$f = \phi\psi\xi, \quad g = \phi\psi\eta,$$

where

$$\begin{aligned} \phi &= u^2 + v^2, & \psi &= u^4 + v^2 + u^2w^2, \\ \xi &= u^4 + v^2 + u^2(w - x)^2, & \eta &= u^4 + (v - xu)^2 + u^2w^2. \end{aligned}$$

**Lemma 1.5.** *The functions  $f$  and  $g$  are almost Blow-analytically equivalent.*

*Proof.* Set  $X = \{0\} \times \mathbf{R}^2 \subset \mathbf{R}^4$ . Let  $\Lambda$  denote the set of half-lines in  $\mathbf{R}^4$  starting from points in  $X$  and orthogonal to  $X$ . Denote by  $e(\lambda)$  the endpoint of  $\lambda \in \Lambda$ , namely  $e(\lambda) = \lambda \cap X$ . Note that the functions  $f|_\lambda$  and  $g|_\lambda$  have singularities only at  $e(\lambda)$  for any  $\lambda \in \Lambda$ , and that  $f(\lambda) = g(\lambda) = [0, \infty)$ .

We will prove the almost Blow-analytic equivalence of  $f$  and  $g$  in such a way that the diffeomorphism that will realize the equivalence, induces a diffeomorphism of  $\mathbf{R}^4 - X$  carrying any  $\lambda - e(\lambda)$ , with  $\lambda \in \Lambda$ , to some  $\lambda' - e(\lambda')$ , with  $\lambda' \in \Lambda$ .

Let  $\pi : M \rightarrow \mathbf{R}^4$  denote the blowing-up along center  $X$ . Then

$$\begin{aligned} M &= \{(s : t, u, v, w, x) \in \mathbf{P}(1) \times \mathbf{R}^4 : sv = tu\}, \\ f \circ \pi(1 : t, u, v, w, x) &= u^6(1 + t^2)(u^2 + t^2 + w^2)(u^2 + t^2 + (w - x)^2), \\ g \circ \pi(1 : t, u, v, w, x) &= u^6(1 + t^2)(u^2 + t^2 + w^2)(u^2 + (t - x)^2 + w^2), \\ f \circ \pi(s : 1, u, v, w, x) &= v^6(1 + s^2)(s^4v^2 + 1 + s^2w^2)(s^4v^2 + 1 + s^2(w - x)^2), \\ g \circ \pi(s : 1, u, v, w, x) &= v^6(1 + s^2)(s^4v^2 + 1 + s^2w^2)(s^4v^2 + (1 - xs)^2 + s^2w^2) \end{aligned}$$

and for each  $\lambda \in \Lambda$ , the set  $\pi^{-1}(\lambda - e(\lambda))$  is defined by

$$(*) \quad \{(1 : t_0, u, t_0u, w_0, x_0) : u \in (0, \infty)\}$$

or

$$\{(s_0 : 1, s_0v, v, w_0, x_0) : v \in (0, \infty)\},$$

depending on the chart, for some  $s_0, t_0, w_0, x_0 \in \mathbf{R}$ . Hence

$$\text{Sing } f \circ \pi = \text{Sing } g \circ \pi = \pi^{-1}(X)$$

and the germs of  $f \circ \pi$  and  $g \circ \pi$  at points of  $\pi^{-1}(X) - M_1 - M_2$  and  $\pi^{-1}(X) - M_1 - M_3$ , respectively, are sixth powers of some regular function germs, where

$$\begin{aligned} M_1 &= \{(1 : 0, 0, 0, 0, x) \in M\}, \\ M_2 &= \{(1 : 0, 0, 0, w, x) \in M : w = x\}, \\ M_3 &= \{(1 : t, 0, 0, 0, x) \in M : t = x\}. \end{aligned}$$

We will construct below a  $C^\infty$  diffeomorphism  $h$  of  $M$  such that  $f \circ \pi$  is equal to  $g \circ \pi \circ h$ , the image of  $M_1 \cup M_2$  under  $h$  is equal to  $M_1 \cup M_3$  and  $h$  is of class  $C^\omega$  on a neighborhood of  $M_1 \cup M_2$ . Assuming the existence of such a diffeomorphism, we obtain by easy calculations two compositions of two sequences of blowings-up  $\pi_f : N \rightarrow M$  and  $\pi_g : L \rightarrow M$  along smooth analytic centers and a  $C^\infty$  diffeomorphism  $\tilde{h} : N \rightarrow L$  such that the union of centers in  $M$  and the images of the centers is  $M_1 \cup M_2 \cup M_3$ ,  $f \circ \pi \circ \pi_f = g \circ \pi \circ \pi_g \circ \tilde{h}$  and the functions  $f \circ \pi \circ \pi_f$  and  $g \circ \pi \circ \pi_g$  have only normal crossing singularities. Therefore  $f \circ \pi \circ \pi_f$  and  $g \circ \pi \circ \pi_g$  are  $C^\omega$  right equivalent by Theorem 3.1,(1) in [6], i.e.  $f$  and  $g$  are almost Blow-analytically equivalent.

Now we give the construction of the expected  $C^\infty$  diffeomorphism  $h$ . Denote by  $(t, w)$  the variables of  $\mathbf{R}^2$ , and for each  $x \in \mathbf{R}$ , let  $B_{|x|}$  denote the ball in  $\mathbf{R}^2$  with center 0 and radius  $|x|$ . Let  $h_x$  be a  $C^\infty$  diffeomorphism of  $\mathbf{R}^2$  which is identical outside of  $B_{2+|x|}$ , with  $h_x(t, w) = (w, -t)$  on  $B_{1+|x|}$  and such that the map

$$\mathbf{R}^3 \ni (t, w, x) \rightarrow h_x(t, w) \in \mathbf{R}^2$$

is of class  $C^\infty$ . Define the map  $h : \mathbf{R}^4 \rightarrow \mathbf{R}^4$  by  $h(u, t, w, x) = (u, h_x(t, w), x)$ . Then  $h$  is a  $C^\infty$  diffeomorphism. Regarding  $\mathbf{R}^4$  as a subset of  $M$  by the map

$$(u, t, w, x) \rightarrow (1 : t, u, tu, w, x),$$

we consider  $h$  defined on  $\mathbf{R}^4$  in  $M$  and we extend it to the whole of  $M$  by the identity map. Therefore we obtain a  $C^\infty$  diffeomorphism  $\tilde{h}$  of  $M$  such that

- $\tilde{h}(\pi^{-1}(X)) = \pi^{-1}(X)$ ,
- the image  $U$  of  $\mathbf{R} \times \cup_{x \in \mathbf{R}} (B_{1+|x|} \times \{x\})$  in  $M$  under the inclusion map is a neighborhood of  $M_1 \cup M_2 \cup M_3$ ,
- $\tilde{h}$  is of class  $C^\omega$  on  $U$ ,
- $\tilde{h}(M_1) = M_1$ ,
- $\tilde{h}(M_2) = M_3$ ,
- for any  $\lambda \in \Lambda$  there exists  $\lambda'$  by  $(*)$  such that  $\tilde{h}(\pi^{-1}(\lambda - e(\lambda))) = \pi^{-1}(\lambda' - e(\lambda'))$ ,
- $\tilde{h}$  is of class  $C^\omega$  on  $U$ ,
- and  $f \circ \pi / (1 + t^2) = g \circ \pi \circ \tilde{h} / (1 + w^2) = u^6(u^2 + t^2 + w^2)(u^2 + t^2 + (w - x)^2)$  on  $U$ .

We modify  $\tilde{h}$  so that  $f \circ \pi = g \circ \pi \circ \tilde{h}$  on  $U$  as follows. Set

$$h_1(u, t, w, x) = (1 + w^2)^{1/10}(u, t, w, x),$$

$$h_2(u, t, w, x) = (1 + t^2)^{1/10}(u, t, w, x),$$

let  $h_3$  be a  $C^\infty$  diffeomorphism of  $\mathbf{R}^4$  which equals  $h_1^{-1} \circ h_2$  on  $\mathbf{R} \times \cup_{x \in \mathbf{R}} (B_{\alpha(x)} \times \{x\})$  and the identity map outside of  $\mathbf{R} \times \cup_{x \in \mathbf{R}} (B_{\beta(x)} \times \{x\})$  for some positive  $C^\infty$  functions  $\alpha$  and  $\beta$  on  $\mathbf{R}$ , and extend  $h_3$  to a  $C^\infty$  diffeomorphism  $\tilde{h}_3$  of  $M$  as above. Then we can choose  $h_1$ ,  $h_2$ ,  $\alpha$  and  $\beta$  so that  $\tilde{h} \circ \tilde{h}_3$  satisfies the above conditions on  $\tilde{h}$  except the last, and the last becomes

$$f \circ \pi = g \circ \pi \circ \tilde{h} \circ \tilde{h}_3 \quad \text{on } U$$

We replace  $\tilde{h}$  with  $\tilde{h} \circ \tilde{h}_3$  and write  $\tilde{h} \circ \tilde{h}_3$  as  $\tilde{h}$ . Thus we have  $f \circ \pi = g \circ \pi \circ \tilde{h}$  on  $U$ .

It remains only to modify  $\tilde{h}$  outside of  $U$  so that the equality  $f \circ \pi = g \circ \pi \circ \tilde{h}$  holds everywhere. For any  $a \in M$ , define  $\tilde{\tilde{h}}(a)$  by

$$\tilde{\tilde{h}}(a) = \overline{\pi^{-1}(\lambda - e(\lambda))} \cap (g \circ \pi)^{-1}(f \circ \pi(a))$$

for some  $\lambda \in \Lambda$  satisfying  $\tilde{h}(a) \in \overline{\pi^{-1}(\lambda - e(\lambda))}$ . Then  $\tilde{\tilde{h}}$  is a well-defined  $C^\infty$  diffeomorphism of  $M$  since

$$\tilde{\tilde{h}}(a) = \overline{\pi^{-1}(\lambda - e(\lambda))} \cap (g \circ \pi)^{-1}(f \circ \pi(a))$$

for  $a \in U$ , the set  $U$  is a union of some  $\overline{\pi^{-1}(\lambda - e(\lambda))}$ ,  $\lambda \in \Lambda$ , and because the restrictions of  $f \circ \pi$  and  $g \circ \pi$  to each  $\overline{\pi^{-1}(\lambda - e(\lambda))}$  outside of  $U$  are  $C^\infty$  right equivalent to the function

$$[0, \infty) \ni z \rightarrow z^6 \in \mathbf{R}.$$

Now the equality  $f \circ \pi = g \circ \pi \circ \tilde{\tilde{h}}$  is satisfied on  $M$ . □

**Lemma 1.6.** *The germs of  $f$  and  $g$  at the origin are not blow-analytically equivalent.*

We have to prove that there do not exist open neighborhoods  $U$  and  $V$  of the origin in  $\mathbf{R}^4$  such that  $f|_U$  and  $g|_V$  are blow-analytically equivalent and the induced homeomorphism from  $U$  to  $V$  fixes the origin.



*Proof.* We introduce an invariant of blow-analytic equivalence, which is a generalisation of one in [8]. Let  $C$  denote the set of germs at the origin of analytic curves  $c : [0, \epsilon) \rightarrow \mathbf{R}^4$ , with  $\epsilon > 0$ . We give a topology on  $C$  by identifying  $C$  with 4-product of the one-variable convergent power series ring  $\mathbf{R}\langle\langle t \rangle\rangle^4$  and choosing the product topology on  $\mathbf{R}\langle\langle t \rangle\rangle$ , i.e.

$$\mathbf{R}\langle\langle t \rangle\rangle \ni \sum_{n=0}^{\infty} a_{n,k} t^n \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

if for any  $n = 0, \dots, \infty$

$$\mathbf{R} \ni a_{n,k} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Then  $p : C \rightarrow \mathbf{R}^4$ , defined by  $p(c) = c(0)$ , is a topological fibre bundle.

Let  $\pi : M \rightarrow \mathbf{R}^4$  be the composition of a finite sequence of blowings-up of  $\mathbf{R}^4$  along smooth analytic centres. The map  $\pi$  naturally induces a surjective  $C^0$  map  $\pi_* : C_M \rightarrow C$  such that  $p \circ \pi_* = \pi \circ p_M$ , where  $C_M$  is the analytic curve germs in  $M$  and  $p_M : C_M \rightarrow M$  is defined by  $p_M(c) = c(0)$ .

Let us assume that the germs of  $f$  and  $g$  at 0 are blow-analytically equivalent. Then there exist open neighbourhoods  $U$  and  $V$  of 0 in  $\mathbf{R}^4$  and a map

$$\tau : \{c \in C : c(0) \in U, f \circ c \not\equiv 0\} \rightarrow \{c \in C : c(0) \in V, g \circ c \not\equiv 0\}$$

such that  $f \circ c = g \circ \tau(c)$  as analytic function germs in the variable  $t$  for  $c \in p^{-1}(U)$ . Let  $\tau_0 : U \rightarrow V$  denote the homeomorphism such that  $p \circ \tau = \tau_0 \circ p$ .

We denote by  $o_f(c)$  the order of  $f \circ c$  at 0 for  $c \in C$ , so that we define a map  $o_f : C \rightarrow \mathbf{N} \cup \{\infty\}$ . Set  $C_a = p^{-1}(a)$ , for  $a \in U$ , and consider the family  $\{C_a \cap o_f^{-1}(i) : i \in \mathbf{N}\}$ . We stratify  $U$  by the groups of the family. Set

$$X_1 = \{a \in U : o_f = 0 \text{ on } C_a\}$$

Then

$$X_1 = \{(u, v, w, x) \in U : u \neq 0 \text{ or } v \neq 0\}.$$

Set also

$$\begin{aligned} X_2 &= \{u = v = 0, w \neq 0, w \neq x\}, \\ X_3 &= \{u = v = w = 0, x \neq 0\} \cup \{u = v = 0, w = x \neq 0\}, \\ X_4 &= \{u = v = w = x = 0\}. \end{aligned}$$

Then  $\{X_1, X_2, X_3, X_4\}$  is a stratification of  $U$ . Moreover if  $a$  belongs to  $X_2$ , then  $(C_a, C_a \cap o_f^{-1}(6))$  has the same homotopy groups as  $([-1, 1]^2, \partial([-1, 1]^2))$  because

$$C_a \cap o_f^{-1}(6) = \{c = (c_1, \dots, c_4) : [0, \epsilon) \rightarrow \mathbf{R}^4 \in C_a : \frac{dc_1}{dt}(0) \neq 0 \text{ or } \frac{dc_2}{dt}(0) \neq 0\}.$$

Similarly if  $a$  belongs to  $X_3$  or  $X_4$ , then  $(C_a, C_a \cap o_f^{-1}(6))$  has the same homotopy groups as  $([-1, 1]^2, (\partial[-1, 1]^2)^2)$  because

$$C_a \cap o_f^{-1}(6) = \{\frac{dc_1}{dt}(0) \neq 0, \frac{dc_2}{dt}(0) \neq 0\}.$$

Now we consider  $g$ . Define  $o_g$  in the same way and set

$$Y_1 = \{a \in V : o_g = 0 \text{ on } C_a\} = \{(u, v, w, x) \in V : u \neq 0 \text{ or } v \neq 0\},$$

$$\begin{aligned} Y_2 &= \{u = v = 0, w \neq 0\}, \\ Y_3 &= \{u = v = w = 0, x \neq 0\}, \\ Y_4 &= \{u = v = w = x = 0\}. \end{aligned}$$

If  $a \in Y_1, Y_2, Y_3$  or  $Y_4$ , then  $(C_a, C_a \cap o_g^{-1}(6))$  has the same homotopy groups as  $([-1, 1]^2, \emptyset)$ ,  $([-1, 1]^2, \partial([-1, 1]^2))$ ,  $([-1, 1]^2, \{6 \text{ points in } \partial([-1, 1]^2)\})$  or  $([-1, 1]^2, (\partial([-1, 1]^2))^2)$  respectively. We omit its proof except for  $Y_3$  because the other case is treated in the same way as above. Let  $a \in Y_3$ . Then

$$C_a \cap o_g^{-1}(6) = \left\{ \frac{dc_1}{dt}(0) \neq 0, \frac{dc_2}{dt}(0) \neq 0, x \frac{dc_1}{dt}(0) \neq \frac{dc_2}{dt}(0) \right\}.$$

Hence  $(C_a, C_a \cap o_g^{-1}(6))$  has the same homotopy groups as  $([-1, 1]^2, \{6 \text{ points in } \partial([-1, 1]^2)\})$ . Note in any case  $\frac{dc_1}{dt}(0) \neq 0$  or  $\frac{dc_2}{dt}(0) \neq 0$  for  $c = (c_1, \dots, c_4) \in C_a \cap o_g^{-1}(6)$ . Thus  $\{Y_1, Y_2, Y_3, Y_4\}$  is a stratification of  $V$  by the homotopy groups of  $C_a \cap o_g^{-1}(6)$ .

On the other hand,  $o_f(c)$  is equal to  $o_g(\tau(c))$  for any  $c \in C$  with  $p(c) \in U$  and  $o_f(c) < \infty$ , since  $f$  and  $g$  are blow-analytically equivalent. Hence for  $a \in U$ , the maps  $o_f : \{c \in C_a : o_f(c) < \infty\} \rightarrow \mathbf{N}$  and  $o_g : \{c \in C_{\tau_0(a)} : o_g(c) < \infty\} \rightarrow \mathbf{N}$  are  $C^0$  right equivalent. Moreover, the restriction of  $\tau$  to  $\{c \in C_a : o_f(c) = 6\}$  is a homeomorphism onto  $\{c \in C_{\tau_0(a)} : o_g(c) = 6\}$ . For that it suffices to prove the following statement.

Let  $M \xrightarrow{\pi_1} M_1 \rightarrow \dots \xrightarrow{\pi_k} M_k = \mathbf{R}^4$  be a sequence of blowings-up along smooth analytic centers such that the images of centers in  $\mathbf{R}^4$  are included in  $\{u = v = 0\}$ . Set  $\pi = \pi_k \circ \dots \circ \pi_1$ . Then the following maps are homeomorphisms:

$$\begin{aligned} \pi_*^{-1}\{c \in C_a : o_f(c) = 6\} &\xrightarrow{\pi_{1*}} \dots \xrightarrow{\pi_{k*}} \{c \in C_a : o_f(c) = 6\}, \\ \pi_*^{-1}\{c \in C_{\tau_0(a)} : o_g(c) = 6\} &\xrightarrow{\pi_{1*}} \dots \xrightarrow{\pi_{k*}} \{c \in C_{\tau_0(a)} : o_g(c) = 6\}. \end{aligned}$$

Consider only the maps in the former sequence. They are clearly continuous and bijective. Hence we show the inverse maps are continuous. Let  $c = (c_1, \dots, c_4) \in C_a$  with  $o_f(c) = 6$ . Then  $\frac{dc_1}{dt}(0) \neq 0$  or  $\frac{dc_2}{dt}(0) \neq 0$  by the above arguments. Namely  $c$  is transverse to  $\{u = v = 0\}$  and hence to the center of  $\pi_k$ . Hence the inverse map of the map  $\pi_{k*}^{-1}\{c \in C_a : o_f(c) = 6\} \rightarrow \{c \in C_a : o_f(c) = 6\}$  is continuous by the definition of a blowing-up. It follows also that for that  $c$ ,  $\pi_{k*}^{-1}(c)$  is transverse to each irreducible component of  $\pi_k^{-1}\{u = v = 0\}$ . Thus repeating the same arguments we see the inverse maps are continuous.

Therefore,  $\{c \in C_a : o_f(c) = 6\}$  and  $\{c \in C_{\tau_0(a)} : o_g(c) = 6\}$  have the same homotopy groups. Hence

$$\tau_0(X_1) = Y_1, \tau_0(X_2) = Y_2, \tau_0(X_3 \cup X_4) = Y_4,$$

which contradicts the assumption that  $\tau_0$  is a homeomorphism.  $\square$

**Remark 1.7.** We can prove moreover that  $f$  and  $g$  are almost Blow-Nash equivalent. Actually, similarly to the preceding proof, we can find compositions of finite sequences of blowings-up of  $\mathbf{R}^4$  along smooth Nash centers  $\pi_f : N \rightarrow \mathbf{R}^4$  and  $\pi_g : L \rightarrow \mathbf{R}^4$  so that  $f \circ \pi_f$  and  $g \circ \pi_g$  are semi-algebraically  $C^m$  right equivalent for any  $m \in \mathbf{N}$  by using a partition of unity of class semi-algebraic  $C^m$ , § II.2, [20]. We may conclude the statement by theorem 3.1.(3) in [6].

We have finally proved:

**Proposition 1.8.** *Almost Blow-analytically equivalent function germs are not necessarily blow-analytically equivalent.*

**1.3. Normal crossings and cardinality.** An analytic function with *only normal crossing singularities* at a point  $x$  of a manifold is a function whose germ at  $x$  is of the form

$$\pm x^\alpha (= \pm \prod_{i=1}^n x_i^{\alpha_i}) + \text{const}, \quad \alpha = (\alpha_1, \dots, \alpha_n) \neq 0 \in \mathbf{N}^n$$

for some local analytic coordinate system  $(x_1, \dots, x_n)$  at  $x$ . If the function has only normal crossing singularities everywhere, we say the function has *only normal crossing singularities*. By Hironaka Desingularization Theorem, any analytic function becomes one with only normal crossing singularities after a finite sequence of blowings-up along smooth centers in the local case or in the compact case. An analytic subset of an analytic manifold is called *normal crossing* if it is the zero set of an analytic function with only normal crossing singularities. This analytic function is called *defined by* the analytic set. It is not unique. However, the sheaf of  $\mathcal{O}$ -ideals *defined by* the analytic set is naturally defined and unique. We can naturally stratify a normal crossing analytic subset  $X$  into analytic manifolds  $X_i$  of dimension  $i$ . We call  $\{X_i\}$  the *canonical stratification* of  $X$ .

The difference between almost Blow-analytic equivalence and almost Blow-analytic R-L equivalence is tiny.

**Proposition 1.9.** *If the germs of  $f$  and  $g$  on a compact connected component of  $\text{Sing } f$  in  $M$  are almost Blow-analytically R-L equivalent then the germs of  $f$  and  $g + \text{const}$  or of  $f$  and  $-g + \text{const}$  are almost Blow-analytically equivalent.*

*Proof.* We can reduce the problem to the case where  $f \circ \pi_f$  and  $g \circ \pi_g$  have only normal crossing singularities by Hironaka Desingularization Theorem. Then the result follows from the remark following lemma 4.7 and proposition 4.8,(i) in [6].  $\square$

Let  $P$  denote the set of homogeneous polynomial functions on  $\mathbf{R}^2$  of degree 4. It is easy to see that the analytic R-L equivalence classes of  $P$  has the cardinality of the continuum, and the Blow-analytic R-L equivalence classes of  $P$  is finite. Moreover almost Blow-analytic R-L equivalence on analytic functions on  $\mathbf{R}^2$  coincides with Blow-analytic R-L equivalence and is an equivalence relation.

For general dimension, we do not know whether almost Blow-analytic (Blow-Nash) R-L equivalence is an equivalence relation, as noticed before. Therefore we will say that analytic (Nash) functions  $f$  and  $g$  lie in the *same class* if there exists a sequence of analytic functions  $f_0, \dots, f_k$  such that  $f_0 = f$ ,  $f_k = g$  and  $f_i$  and  $f_{i+1}$  are almost Blow-analytically (Blow-Nash) R-L equivalent,  $i = 0, \dots, k-1$ .

In that setting, we are able to determine the cardinality of the classes under these relations.

**Theorem 1.10.** *Let  $M$  be a compact analytic (resp. Nash) manifold of strictly positive dimension. Then the cardinality of the set of classes of analytic (resp. Nash) functions on  $M$ , classified by almost Blow-analytic (resp. Blow-Nash) R-L equivalence, is countable. Moreover, in the Nash case, we do not need to assume that  $M$  is compact.*

We proved in [6] that the cardinality of analytic R-L equivalence classes of analytic functions on  $M$  with only normal crossing singularities is 0 or countable. In particular, it can not be 0 in theorem 1.11 since we can produce non-trivial examples. We proved moreover in [6] that in the Nash case, the result holds true even if  $M$  is no longer compact, and in that case the cardinality is always countable. Combined with Hironaka Desingularization Theorem, we obtain a proof for theorem 1.11.

**1.4. Nash Approximation Theorems.** The approximation of analytic solutions of a system of Nash equations is a crucial tool in the study of real analytic and Nash functions on Nash manifolds. A local version is given by the classical Artin approximation theorem. The Nash Approximation Theorem of [3] states a global version on a compact Nash manifold.

The following result is a natural counterpart, for almost Blow-analytic equivalence, of the Nash Approximation Theorem of [3]. The remaining part of the paper will be devoted to its proof.

**Theorem 1.11.** *Let  $M$  be a Nash manifold,  $X \subset M$  be a compact semialgebraic subset and  $f, g$  Nash function germs on  $X$  in  $M$  such that  $X = M$  or  $X \subset \text{Sing } f$ . If  $f$  and  $g$  are almost Blow-analytically (R-L) equivalent, then  $f$  and  $g$  are almost Blow-Nash (respectively R-L) equivalent.*

**Remark 1.12.**

- (1) Here the compactness assumption on  $X$  is necessary. Indeed, there exist a non-compact Nash manifold  $M$  and Nash functions  $f$  and  $g$  on  $M$  which are  $C^\omega$  right equivalent but not almost Blow-Nash equivalent as follows. Let  $N$  be a compact contractible Nash manifold with non-simply connected boundary of dimension  $n > 3$  (e.g., see [16]). Set  $M = (\text{Int } N) \times (0, 1)$  and let  $f : M \rightarrow (0, 1)$  denote the projection. Then  $M$  and  $f$  are of class Nash, and  $M$  is Nash diffeomorphic to  $\mathbf{R}^{n+1}$  for the following reason. Smooth the corners of  $N \times [0, 1]$ . Then  $N \times [0, 1]$  is a compact contractible Nash manifold with simply connected boundary of dimension  $> 4$ . Hence by the positive answers to Poincaré conjecture and Schönflies problem (Brown-Mazur Theorem)  $N \times [0, 1]$  is  $C^\infty$  diffeomorphic to an  $(n + 1)$ -ball. Hence by Theorem VI.2.2, [20]  $M$  is Nash diffeomorphic to an open  $(n + 1)$ -ball. Let  $g : M \rightarrow \mathbf{R}$  be a Nash function which is Nash right equivalent to the projection  $\mathbf{R}^n \times (0, 1) \rightarrow (0, 1)$ . Then  $f$  and  $g$  are  $C^\omega$  right equivalent since  $\text{Int } N$  is  $C^\omega$  diffeomorphic to  $\mathbf{R}^n$ , but they are not almost Blow-Nash equivalent because if they were so then their levels would be Nash diffeomorphic except for a finite number of values and hence  $\text{Int } N$  and  $\mathbf{R}^n$  are Nash diffeomorphic, which contradicts Theorem VI 2.2, [20].
- (2) The similar result concerning Blow-Nash equivalence remains open. Namely we do not know whether Blow-analytically equivalent Nash function germs on  $X$  in  $M$  are Blow-Nash equivalent.

## 2. NASH APPROXIMATION OF AN ANALYTIC DESINGULARIZATION

**2.1. Preliminaries on real analytic sheaf theory.** We recall the statements of the real analytic case of Cartan Theorems A and B, and Oka Theorem, in the refined version given in [6].

Let  $\mathcal{O}$ ,  $\mathcal{N}$  and  $N(M)$  denote, respectively, the sheaves of analytic and Nash function germs on an analytic and Nash manifold and the ring of Nash functions on a Nash manifold  $M$ . We write  $\mathcal{O}_M$  and  $\mathcal{N}_M$  when we emphasize the domain  $M$ . For a function  $f$  on an analytic (Nash) manifold  $M$ , a subset  $X$  of  $M$ , a vector field  $v$  on  $M$  and for a sheaf of  $\mathcal{O}$ -( $\mathcal{N}$ -) modules  $\mathcal{M}$  on  $M$ , let  $f_x$ ,  $X_x$ ,  $v_x$  and  $\mathcal{M}_x$  denote the germs of  $f$  and  $X$  at a point  $x$  of  $M$ , the tangent vector assigned to  $x$  by  $v$  and the stalk of  $\mathcal{M}$  at  $x$ , respectively. For a compact semialgebraic subset  $X$  of a Nash manifold  $M$ , let  $\mathcal{N}(X)$  denote the germs of Nash functions on  $X$  in  $M$  with the topology of the inductive limit space of the topological spaces  $N(U)$  with the compact-open  $C^\infty$  topology where  $U$  runs through the family of open semialgebraic neighborhoods of  $X$  in  $M$ .

**Theorem 2.1.** *Let  $\mathcal{M}$  be a coherent sheaf of  $\mathcal{O}$ -modules on an analytic manifold  $M$ .*

- (1) (Cartan Theorem A) *For any  $x \in M$  we have  $\mathcal{M}_x = H^0(M, \mathcal{M})\mathcal{O}_x$ .*
- (2) *Assume moreover that  $\mathcal{M}_x$  is generated by a uniform number of elements for any  $x \in M$ . Then  $H^0(M, \mathcal{M})$  is finitely generated as an  $H^0(M, \mathcal{O})$ -module.*
- (3) (Cartan Theorem B)  $H^1(M, \mathcal{M}) = 0$ .
- (4) *Let  $X \subset M$  be a global analytic set—the zero set of an analytic function. Let  $\mathcal{I}$  be a coherent sheaf of  $\mathcal{O}$ -ideals on  $M$  such that any element of  $\mathcal{I}$  vanishes on  $X$ . Then any  $f \in H^0(M, \mathcal{O}/\mathcal{I})$  can be extended to some  $F \in C^\omega(M)$ , i.e.  $f$  is the image of  $F$  under the natural map  $H^0(M, \mathcal{O}) \rightarrow H^0(M, \mathcal{O}/\mathcal{I})$ . If  $X$  is normal crossing, we can choose  $\mathcal{I}$  to be the function germs vanishing on  $X$ . Then  $H^0(M, \mathcal{O}/\mathcal{I})$  consists of*

functions on  $X$  whose germs at each point of  $X$  are extensible to analytic function germs on  $M$ .

- (5) (*Oka Theorem*) Let  $\mathcal{M}_1$  and  $\mathcal{M}_2$  be coherent sheaves of  $\mathcal{O}$ -modules on  $M$ , and  $h : \mathcal{M}_1 \rightarrow \mathcal{M}_2$  be an  $\mathcal{O}$ -homomorphism. Then  $\text{Ker } h$  is a coherent sheaf of  $\mathcal{O}$ -modules.

**2.2. Euclidean realization of a sequence of blowings-up.** Let  $C$  be a smooth analytic subset of an analytic manifold  $U$ , and let  $\pi : M \rightarrow U$  denote the blowing-up of  $U$  along center  $C$ . In this section, we describe  $M$  as a smooth analytic subset of  $U \times \mathbf{P}(k)$  for some  $k \in \mathbf{N}$ .

Let  $\mathcal{I}$  denote the sheaf of  $\mathcal{O}$ -ideals defined by  $C$ . Since  $C$  is smooth, each stalk  $\mathcal{I}_x$  is generated by  $c = \text{codim } C$  elements. Hence there exist a finite number of global generators  $h_0, \dots, h_k \in H^0(U, \mathcal{I})$  of  $\mathcal{I}$  by theorem 2.1.(2). Define  $\mathcal{A}$  to be the sheaf of relations of  $h_0, \dots, h_k$ :

$$\mathcal{A} = \cup_{x \in U} \{(\mu_0, \dots, \mu_k) \in \mathcal{O}_x^{k+1} : \sum_{i=0}^k \mu_i h_{ix} = 0\}.$$

Then  $\mathcal{A}$  is coherent by theorem 2.1.(5), and each  $\mathcal{A}_x$  is generated by  $k - c + 1 + (c - 1)!$  elements as follows. If  $x_0 \notin C$  then  $h_i(x_0) \neq 0$  for some  $i$ , say 0. On a small neighborhood of  $x_0$ , the map  $\mathcal{O}^k \supset \mathcal{O}_x^k \ni (\mu_1, \dots, \mu_k) \rightarrow (-\sum_{i=1}^k \mu_i h_{ix}/h_{0x}, \mu_1, \dots, \mu_k) \in \mathcal{O}_x^{k+1} \subset \mathcal{O}^{k+1}$  is an isomorphism onto  $\mathcal{A}$ . Hence  $\mathcal{A}_x$  is generated by  $k$  elements. If  $x_0 \in C$ , let  $x$  denote a point near  $x_0$ . In this case we can assume that  $h_{0x}, \dots, h_{c-1x}$  are regular function germs and generate  $\mathcal{I}_x$ . Then each  $h_{ix}$ ,  $c \leq i \leq k$ , is of the form  $\sum_{i=0}^{c-1} \phi_i h_{ix}$  for some  $\phi_i \in \mathcal{O}_x$ . Hence the projection image of  $\mathcal{A}_x$  to the last  $k - c + 1$  factors of  $\mathcal{O}_x^{k+1}$  is  $\mathcal{O}_x^{k-c+1}$ , and it suffices to see that  $\mathcal{A}_x \cap \mathcal{O}_x^c \times \{0\} \times \dots \times \{0\}$  is generated by  $(c - 1)!$  elements. We do this as follows:

$\mathcal{A}_x \cap \mathcal{O}_x^c \times \{0\} \times \dots \times \{0\}$  is generated by  $(0, \dots, 0, \overbrace{h_{j-1x}}^i, 0, \dots, -\overbrace{h_{i-1x}}^j, 0, \dots, 0)$  for  $1 \leq i < j \leq c$ . Therefore,  $\mathcal{A}$  is generated by its global cross-sections  $g_1 = (g_{1,0}, \dots, g_{1,k}), \dots, g_{k'} = (g_{k',0}, \dots, g_{k',k}) \in C^\omega(U)^{k+1}$  for some  $k' \in \mathbf{N}$ . Moreover, it follows from these arguments that

- (1)  $\sum_{j=0}^k g_{i,j} h_j = 0$ ,  $i = 1, \dots, k'$ ,
- (2) for each  $x \in U - C$ , the vectors  $g_1(x), \dots, g_{k'}(x)$  in  $\mathbf{R}^{k+1}$  span a hyperplane and  $(h_0(x), \dots, h_k(x))$  in  $\mathbf{R}^{k+1}$  is non-zero and orthogonal to the hyperplane,
- (3) for each  $x \in C$ , the linear subspace  $\{(s_0, \dots, s_k) \in \mathbf{R}^{k+1} : \sum_{j=0}^k s_j g_{i,j}(x) = 0, i = 1, \dots, k'\}$  of  $\mathbf{R}^{k+1}$  is of dimension  $c$ .

Hence we can regard set-theoretically  $M - \pi^{-1}(C)$  as

$$\{(x, t) \in (U - C) \times \mathbf{P}(k) : t_i h_j(x) = t_j h_i(x), i, j = 0, \dots, k\}$$

by (2), hence  $M$  as

$$\{(x, t) \in U \times \mathbf{P}(k) : t_i h_j(x) = t_j h_i(x), i, j = 0, \dots, k, \text{ and } \sum_{j=0}^k t_j g_{i,j}(x) = 0, i = 1, \dots, k'\}$$

by (3) and by  $\sum_{j=0}^k t_j g_{i,j}(x) = 0$ ,  $i = 1, \dots, k'$ , for  $(x, t) \in (U - C) \times \mathbf{P}(k)$  with  $t_i h_j(x) = t_j h_i(x)$ ,  $i, j = 0, \dots, k$ , and  $\pi$  as the restriction to  $M$  of the projection  $U \times \mathbf{P}(k) \rightarrow U$ . When we identify  $M$  with the subset of  $U \times \mathbf{P}(k)$ , we say  $M$  is *realized* in  $U \times \mathbf{P}(k)$ .

Since we treat only finite sequences of blowings-up, we can embed  $M$  into a Euclidean space. For that we embed  $\mathbf{P}(k)$  algebraically in  $\mathbf{R}^{(k+1)^2}$  as in [2] by

$$(t_0 : \dots : t_k) \mapsto \left( \frac{t_i t_j}{|t|^2} \right),$$

where  $|t|^2 = \sum_{i=0}^k t_i^2$ . It is known that  $\mathbf{P}(k)$  is a non-singular algebraic subvariety in  $\mathbf{R}^{(k+1)^2}$ . We denote by  $y_{i,j}$  the coordinates on  $\mathbf{R}^{(k+1)^2}$  such that  $y_{i,j} = t_i t_j / |t|^2$  on  $\mathbf{P}(k)$ . Let  $\xi_1, \dots, \xi_s$  be generators of the ideal of  $\mathbf{R}[y_{i,j}]$  of functions vanishing on  $\mathbf{P}(k)$ . Set  $l_{i,j,m}(x, y) = y_{i,j} h_m(x) - y_{m,i} h_j(x)$  for  $i, j, m = 0, \dots, k$ . Define

$$N = \{(x, y) \in U \times \mathbf{R}^{(k+1)^2} : l_{i,j,m}(x, y) = 0, \ i, j, m = 0, \dots, k, \\ \sum_{j=0}^k y_{j,m} g_{i,j}(x) = 0, \ i = 1, \dots, k', \ m = 0, \dots, k, \text{ and } \xi_i(y) = 0, \ i = 1, \dots, s\}.$$

Then  $M = N$ . Moreover the analytic sets on both sides coincide algebraically, i.e. the functions  $l_{i,j,m}, \sum_{j=0}^k y_{j,m} g_{i,j}, \xi_i$  generate  $I(M)$ —the ideal of  $C^\omega(U \times \mathbf{R}^{(k+1)^2})$  of functions vanishing on  $M$ . Indeed, by theorem 2.1.(4) the problem is local. If  $x \in U - C$ , the claim locally at  $x$  is clear. Assume that  $x \in C$ , and let  $(x_1, \dots, x_n)$  denote a local coordinate system of  $U$  around  $x$ . As the claim does not depend on the choice of  $\{g_i\}$ , we can assume that  $h_j = x_{j+1}, j = 0, \dots, c-1, h_j = \sum_{i=0}^{c-1} \phi_{i,j} h_i, j = c, \dots, k$ , and  $g_1 = (-\phi_{0,c}, \dots, -\phi_{c-1,c}, 1, 0, \dots, 0), \dots, g_{k-c+1} = (-\phi_{0,k}, \dots, -\phi_{c-1,k}, 0, \dots, 0, 1), g_{k-c+2} = (x_2, -x_1, 0, \dots, 0), \dots, g_{k'} = (0, \dots, 0, x_c, -x_{c-1}, 0, \dots, 0)$  for some  $C^\omega$  functions  $\phi_{i,j}$  on a neighborhood of  $x$  and for  $k' = k - c + 1 + (c-1)!$ . Then

$$(t_0 \cdots t_k) = (t_0 \cdots t_{c-1}) \begin{pmatrix} 1 & 0 & \phi_{0,c} & \cdots & \phi_{0,k} \\ & \ddots & \vdots & & \vdots \\ 0 & 1 & \phi_{c-1,c} & \cdots & \phi_{c-1,k} \end{pmatrix}, \\ \begin{pmatrix} t_0 \\ \vdots \\ t_k \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ \phi_{0,c} & \cdots & \phi_{c-1,c} \\ \vdots & & \vdots \\ \phi_{0,k} & \cdots & \phi_{c-1,k} \end{pmatrix} \begin{pmatrix} t_0 \\ \vdots \\ t_{c-1} \end{pmatrix}.$$

Therefore the matrix  $(t_i, t_j)_{i,j=0,\dots,k}$  is equal to

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ \phi_{0,c} & \cdots & \phi_{c-1,c} \\ \vdots & & \vdots \\ \phi_{0,k} & \cdots & \phi_{c-1,k} \end{pmatrix} \begin{pmatrix} t_0 \\ \vdots \\ t_{c-1} \end{pmatrix} (t_0 \cdots t_{c-1}) \begin{pmatrix} 1 & 0 & \phi_{0,c} & \cdots & \phi_{0,k} \\ & \ddots & \vdots & & \vdots \\ 0 & 1 & \phi_{c-1,c} & \cdots & \phi_{c-1,k} \end{pmatrix}$$

whereas the matrix  $(y_{i,j})_{i,j=0,\dots,k}$  equals

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ \phi_{0,c} & \cdots & \phi_{c-1,c} \\ \vdots & & \vdots \\ \phi_{0,k} & \cdots & \phi_{c-1,k} \end{pmatrix} \begin{pmatrix} y_{0,0} & \cdots & y_{0,c-1} \\ \vdots & & \vdots \\ y_{c-1,0} & \cdots & y_{c-1,c-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & \phi_{0,c} & \cdots & \phi_{0,k} \\ & \ddots & \vdots & & \vdots \\ 0 & 1 & \phi_{c-1,c} & \cdots & \phi_{c-1,k} \end{pmatrix}.$$

Hence we can forget  $h_j$  and  $y_{i,j} = y_{j,i}, i = 0, \dots, k, j = c, \dots, k$ , and we can replace  $N$  with its image under the projection  $U \times \mathbf{R}^{(k+1)^2} \ni (x, y_{i,j}) \rightarrow (x, y_{i,j})_{i,j \leq c-1} \in U \times \mathbf{R}^{c^2}$ .

Considering the realization of  $M$  in  $U \times \mathbf{P}(k)$ , it becomes

$$\widetilde{M} = \{(x, y) \in U \times \mathbf{R}^{c^2} : l_{i,j,m}(x, y) = 0, i, j, m = 0, \dots, c-1,$$

$$\sum_{j=0}^{c-1} y_{j,m} g_{i,j}(x) = 0, i = k-c+2, \dots, k', m = 0, \dots, c-1, \text{ and } \xi'_i(y) = 0, i = 1, \dots, s'\},$$

where  $\xi'_i$  are generators of  $I(\mathbf{P}(c-1)) \subset \mathbf{R}[y_{i,j}]_{i,j \leq c-1}$ . Therefore it suffices to show that  $l_{i,j,m}$ ,  $\sum_{j'=0}^{c-1} y_{j',m} g_{i',j'}$ ,  $\xi'_{i''}$ ,  $i, j, m = 0, \dots, c-1$ ,  $i' = k-c+2, \dots, k'$ ,  $i'' = 1, \dots, s'$ , generate  $I(\widetilde{M})$ . However, by easy calculations we prove that  $l_{i,j,m}$  and  $\xi'_{i''}$  generate  $I(\widetilde{M})$ . (To realize  $M$  in  $U \times \mathbf{P}(k)$  we need the equations  $\sum_{j=0}^k t_j g_{i,j}(x) = 0$ ,  $i = 1, \dots, k-c+1$ , which are equivalent to  $t_c = t_0 \phi_{0,c} + \dots + t_{c-1} \phi_{c-1,c}$ ,  $\dots$ ,  $t_k = t_0 \phi_{0,k} + \dots + t_{c-1} \phi_{c-1,k}$ .)

**2.3. Perturbation of a blowing-up.** When we perturb  $h_i$ ,  $i = 0, \dots, k$ , in the strong Whitney  $C^\infty$  topology, the common zero set  $Z(h_i)$  of  $h_i$ 's may become of smaller dimension than  $C$  and singular, where the strong Whitney  $C^\infty$  topology on  $C^\infty(U)$  is defined to be the topology of the projective limit space of the topological spaces  $C^\infty(U_k)$  with the  $C^\infty$  topology for all compact  $C^\infty$  submanifolds possibly with boundary  $U_k$  of  $U$ . (Note that Whitney Approximation Theorem in [22] holds also in this topology, and we call it Whitney Approximation Theorem.) However, we have

**Lemma 2.2.** *Let  $\tilde{h}_i$ ,  $i = 0, \dots, k$ , and  $\tilde{g}_i = (\tilde{g}_{i,0}, \dots, \tilde{g}_{i,k})$ ,  $i = 1, \dots, k'$ , be  $C^\omega$  functions on  $U$  and  $C^\omega$  maps from  $U$  to  $\mathbf{R}^{k+1}$  close to  $h_i$  and  $g_i$ , respectively, in the strong Whitney  $C^\infty$  topology. Assume that (1)  $\sum_{j=0}^k \tilde{g}_{i,j} \tilde{h}_j = 0$ ,  $i = 1, \dots, k'$ . Then*

- $\tilde{C} = Z(\tilde{h}_i)$  is smooth and of the same dimension as  $C$ ,  $\tilde{h}_0, \dots, \tilde{h}_k$  generate  $I(Z(\tilde{h}_i))$  and  $\tilde{g}_1, \dots, \tilde{g}_{k'}$  are generators of the sheaf of relations  $\tilde{\mathcal{A}}$  of  $\tilde{h}_0, \dots, \tilde{h}_k$ .
- Let  $\pi : M \rightarrow U$  and  $\tilde{\pi} : \tilde{M} \rightarrow U$  denote the blowings-up along centers  $C$  and  $\tilde{C}$ , respectively. Let  $M$  and  $\tilde{M}$  be realized in  $U \times \mathbf{P}(k)$  as in section 2.2. Then there exist analytic diffeomorphisms  $\tau$  of  $U$  and  $\psi : M \rightarrow \tilde{M}$  close to  $\text{id}$  in the strong Whitney  $C^\infty$  topology such that  $\tau(C) = \tilde{C}$  and  $\tilde{\pi} \circ \psi = \tau \circ \pi$ .

*Proof.* The problem in the former half is local and clear around a point outside of  $C$ , and hence we assume that  $h_j = x_{j+1}$ ,  $j = 0, \dots, c-1$ , for a local coordinate system  $(x_1, \dots, x_n)$ , and  $h_j = \sum_{i=0}^{c-1} \phi_{i,j} h_i$ ,  $j = c, \dots, k$ , for some  $C^\omega$  functions  $\phi_{i,j}$  on  $U$ . Then  $Z(\tilde{h}_0, \dots, \tilde{h}_{c-1})$  is smooth and of the same dimension as  $C$ . Hence we need to see that  $\tilde{h}_j$ ,  $j = c, \dots, k$ , are contained in the ideal of  $C^\omega(U)$  generated by  $\tilde{h}_j$ ,  $j = 0, \dots, c-1$ . Choose  $C^\omega$  functions  $\alpha_{i,j}$ ,  $i = 1, \dots, k-c+1$ ,  $j = 1, \dots, k'$ , on  $U$  so that

$$\begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,k'} \\ \vdots & & \vdots \\ \alpha_{k-c+1,1} & \cdots & \alpha_{k-c+1,k'} \end{pmatrix} \begin{pmatrix} g_1 \\ \vdots \\ g_{k'} \end{pmatrix}$$

is of the form

$$\begin{pmatrix} -\phi_{0,1} & \cdots & -\phi_{c-1,1} & 1 & 0 \\ \vdots & & \vdots & & \ddots \\ -\phi_{0,k-c+1} & \cdots & -\phi_{c-1,k-c+1} & 0 & 1 \end{pmatrix}.$$

Set

$$\begin{pmatrix} \tilde{g}'_{1,0} & \cdots & \tilde{g}'_{1,k} \\ \vdots & & \vdots \\ \tilde{g}'_{k-c+1,0} & \cdots & \tilde{g}'_{k-c+1,k} \end{pmatrix} = \begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,k'} \\ \vdots & & \vdots \\ \alpha_{k-c+1,1} & \cdots & \alpha_{k-c+1,k'} \end{pmatrix} \begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_{k'} \end{pmatrix}.$$

Then  $\begin{pmatrix} \tilde{g}'_{1,c} & \cdots & \tilde{g}'_{1,k} \\ \vdots & & \vdots \\ \tilde{g}'_{k-c+1,c} & \cdots & \tilde{g}'_{k-c+1,k} \end{pmatrix}$  is close to  $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ . Hence

$$\begin{pmatrix} \tilde{g}'_{1,c} & \cdots & \tilde{g}'_{1,k} \\ \vdots & & \vdots \\ \tilde{g}'_{k-c+1,c} & \cdots & \tilde{g}'_{k-c+1,k} \end{pmatrix}^{-1} \begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,k'} \\ \vdots & & \vdots \\ \alpha_{k-c+1,1} & \cdots & \alpha_{k-c+1,k'} \end{pmatrix} \begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_{k'} \end{pmatrix}$$

is well-defined and of the form  $\begin{pmatrix} \tilde{g}''_{1,0} & \cdots & \tilde{g}''_{1,c-1} & 1 & 0 \\ \vdots & & \vdots & & \ddots \\ \tilde{g}''_{k-c+1,0} & \cdots & \tilde{g}''_{k-c+1,c-1} & 0 & 1 \end{pmatrix}$ . Now (1) implies

$$\begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_{k'} \end{pmatrix} \begin{pmatrix} \tilde{h}_0 \\ \vdots \\ \tilde{h}_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \text{ Therefore,}$$

$$\begin{pmatrix} \tilde{g}''_{1,0} & \cdots & \tilde{g}''_{1,c-1} & 1 & 0 \\ \vdots & & \vdots & & \ddots \\ \tilde{g}''_{k-c+1,0} & \cdots & \tilde{g}''_{k-c+1,c-1} & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{h}_0 \\ \vdots \\ \tilde{h}_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

and  $\tilde{h}_j = -\sum_{i=0}^{c-1} \tilde{g}''_{j-c+1,i} \tilde{h}_i$ ,  $j = c, \dots, k$ .

We need to see that  $\tilde{g}_1, \dots, \tilde{g}_{k'}$  are generators of  $\tilde{\mathcal{A}}$ . By (1) they are global cross-sections of  $\tilde{\mathcal{A}}$ . We postpone proving  $\tilde{g}_1, \dots, \tilde{g}_{k'}$  generate  $\tilde{\mathcal{A}}$ .

Next we prove the latter half of the lemma. We first define  $\tau$  on  $C$ . The condition on  $\tau|_C$  to be satisfied is  $\tau(C) = \tilde{C}$ . Let  $U \subset \mathbf{R}^N$ , let  $q$  denote the orthogonal projection of a tubular neighborhood of  $U$  in  $\mathbf{R}^N$ , and let  $p : V \rightarrow C$  denote the proper orthogonal projection of a small closed tubular neighborhood of  $C$  in  $U$ . We want to require  $\tau$  to satisfy, moreover,  $p \circ \tau = \text{id}$  on  $C$ . Then  $\tau|_C$  is unique and the problem of finding  $\tau|_C$  is local. Hence we assume as above that  $h_j = x_{j+1}$ ,  $j = 0, \dots, c-1$ , for a local coordinate system  $(x_1, \dots, x_n)$  at each point of  $C$  and  $h_j = \sum_{i=0}^{c-1} \phi_{i,j} h_i$ ,  $j = c, \dots, k$ , for some  $C^\omega$  functions  $\phi_{i,j}$ . Then  $\tau|_C$  is well-defined (cf. proof of lemma 4.2 in [6]), and  $\tau|_C$  is an analytic embedding of  $C$  into  $U$  close to  $\text{id}$  in the strong Whitney  $C^\infty$  topology.

Secondly, we extend  $\tau|_C$  to  $V$  by setting  $\tau(x) = q(\tau \circ p(x) + x - p(x))$  for  $x \in V$ , which is close to  $\text{id}$  in the strong Whitney  $C^\infty$  topology. Moreover, using the extension, we extend  $\tau|_C$  to an analytic diffeomorphism  $\tau$  of  $U$  close to  $\text{id}$  in the same topology by using a partition of unity of class  $C^\infty$  and theorem 2.1.(4), combined with Whitney Approximation Theorem [22].

Lastly, we need to find  $\psi$ . Set  $\tilde{h}_i = h_i \circ \tau^{-1}$  and  $\tilde{g}_i = g_i \circ \tau^{-1}$  and define  $\tilde{M}$  by  $\tilde{h}_i$  and  $\tilde{g}_i$  in  $U \times \mathbf{P}(k)$  and  $\tilde{\pi} : \tilde{M} \rightarrow U$ . Then

$$\tilde{M} = \{(x, t) \in U \times \mathbf{P}(k) : t_i h_j \circ \tau^{-1}(x) = t_j h_i \circ \tau^{-1}(x), i, j = 0, \dots, k, \text{ and}$$

$$\sum_{j=0}^k t_j g_{i,j} \circ \tau^{-1}(x) = 0, i = 1, \dots, k'\},$$

and  $\tilde{\psi} : M \ni (x, t) \rightarrow (\tau(x), t) \in \tilde{M}$  is an analytic diffeomorphism close to  $\text{id}$  in the strong Whitney  $C^\infty$  topology such that  $\tilde{\pi} \circ \tilde{\psi} = \tau \circ \pi$ . Hence we can replace  $h_i$  and  $g_i$  with  $h_i \circ \tau^{-1}$  and  $g_i \circ \tau^{-1}$ , respectively. Thus we assume from the beginning that  $Z(h_i) = Z(\tilde{h}_i)$ . Set  $h'_i = \tilde{h}_i - h_i$ . Then there exist analytic functions  $\chi_{i,j}$ ,  $i, j = 0, \dots, k$ , on  $U$  close to 0 in the topology such that  $h'_i = \sum_{j=0}^k \chi_{i,j} h_j$ ,  $i = 0, \dots, k$ , which is proved, as before, by using a



partition of unity of class  $C^\infty$  and theorem 2.1.(4) combined with Whitney Approximation Theorem.

Set  $A(x) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} + \begin{pmatrix} \chi_{0,0} & \cdots & \chi_{k,0} \\ \vdots & & \vdots \\ \chi_{0,k} & \cdots & \chi_{k,k} \end{pmatrix}$  and define an analytic diffeomorphism  $\psi$

of  $U \times \mathbf{P}(k)$  by  $\psi(x, t) = (x, tA(x))$  for  $(x, t) \in U \times \mathbf{P}(k)$ . Then  $(h_0, \dots, h_k)A = (\tilde{h}_0, \dots, \tilde{h}_k)$  on  $U$ , hence  $\psi(M) = \tilde{M}$ ,  $\tilde{\pi} \circ \psi = \pi$  on  $M$  and  $\psi|_M$  is close to id in the topology, which proves the latter half.

It remains to show that  $\tilde{g}_1, \dots, \tilde{g}_{k'}$  generate  $\tilde{\mathcal{A}}$ , i.e.  $\tilde{g}_1, \dots, \tilde{g}_{k'}$  generate the  $C^\omega(U)$ -module

$\tilde{X}$  defined by  $\tilde{X} = \{\tilde{g} \in (C^\omega(U))^{k+1} : \tilde{g} \begin{pmatrix} \tilde{h}_0 \\ \vdots \\ \tilde{h}_k \end{pmatrix} = 0\}$  by theorem 2.1.(3). As above

we can assume that  $Z(h_i) = Z(\tilde{h}_i)$ . Moreover, we suppose that  $\tilde{h}_i = h_i$  for any  $i$  for the following reason. For the above  $A$  we have  ${}^tA \begin{pmatrix} h_0 \\ \vdots \\ h_k \end{pmatrix} = \begin{pmatrix} \tilde{h}_0 \\ \vdots \\ \tilde{h}_k \end{pmatrix}$ . Hence  $\tilde{X} = \{\tilde{g} \in$

$(C^\omega(U))^{k+1} : \tilde{g} {}^tA \begin{pmatrix} h_0 \\ \vdots \\ h_k \end{pmatrix} = 0\}$ . Then it suffices to see that  $\tilde{g}_1 {}^tA, \dots, \tilde{g}_{k'} {}^tA$  generate the

$C^\omega(U)$ -module  $X = \{g \in (C^\omega(U))^{k+1} : g \begin{pmatrix} h_0 \\ \vdots \\ h_k \end{pmatrix} = 0\}$  because the map  $(C^\omega(U))^{k+1} \ni \tilde{g} \rightarrow$

$\tilde{g} {}^tA \in (C^\omega(U))^{k+1}$  is an isomorphism as  $C^\omega(U)$ -modules. Here  $\tilde{g}_1 {}^tA, \dots, \tilde{g}_{k'} {}^tA$  are close to  $g_1, \dots, g_{k'}$  respectively. Therefore, replacing  $\begin{pmatrix} \tilde{h}_0 \\ \vdots \\ \tilde{h}_k \end{pmatrix}$  and  $\begin{pmatrix} \tilde{g}_0 \\ \vdots \\ \tilde{g}_{k'} \end{pmatrix}$  with  $\begin{pmatrix} h_0 \\ \vdots \\ h_k \end{pmatrix}$  and  $\begin{pmatrix} \tilde{g}_0 \\ \vdots \\ \tilde{g}_{k'} \end{pmatrix} {}^tA$ ,

respectively, we suppose from the beginning that  $\tilde{h}_i = h_i$  for all  $i$  and  $\tilde{\mathcal{A}} = \mathcal{A}$ .

As above, the problem is local at each point of  $C$  and we assume that  $h_j = x_{j+1}$ ,  $j = 0, \dots, c-1$ , for a local coordinate system  $(x_1, \dots, x_n)$ . Recall that

$$(*) \quad \begin{pmatrix} \beta_{1,1} & \cdots & \beta_{1,k'} \\ \vdots & & \vdots \\ \beta_{k-c+1,1} & \cdots & \beta_{k-c+1,k'} \end{pmatrix} \begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_{k'} \end{pmatrix} = \begin{pmatrix} \cdot & \cdots & \cdot & 1 & 0 \\ \vdots & & \vdots & & \ddots \\ \cdot & \cdots & \cdot & 0 & 1 \end{pmatrix}$$

for some  $C^\omega$  functions  $\beta_{i,j}$  on  $U$ . Let  $r$  be the restriction to  $\mathcal{A}$  of the projection of  $\mathcal{O}^{k+1}$  to the last  $k-c+1$  factors and let  $r_* : H^0(U, \mathcal{A}) \rightarrow (C^\omega(U))^{k-c+1}$  denote the induced map. Then  $(*)$  implies that  $r_*(\tilde{g}_1), \dots, r_*(\tilde{g}_{k'})$  generate  $r(\mathcal{A}) = \mathcal{O}^{k-c+1}$ . Hence it suffices to see that  $\mathcal{A} \cap \mathcal{O}^c \times \{0\} \times \cdots \times \{0\}$  is generated by  $\{\sum_{i=1}^{k'} \beta_i \tilde{g}_i : \beta_i \in C^\omega(U), r_*(\sum_{i=1}^{k'} \beta_i \tilde{g}_i) = 0\}$ .

Since  $g_1, \dots, g_{k'}$  generate  $\mathcal{A}$ , there exist  $C^\omega$  functions  $\gamma_{i,j}$ ,  $i = 1, \dots, (c-1)!$ ,  $j = 1, \dots, k'$ , such that

$$\begin{pmatrix} \gamma_{1,1} & \cdots & \gamma_{1,k'} \\ \vdots & & \vdots \\ \gamma_{(c-1)!,1} & \cdots & \gamma_{(c-1)!,k'} \end{pmatrix} \begin{pmatrix} g_1 \\ \vdots \\ g_{k'} \end{pmatrix} = \begin{pmatrix} x_2 & -x_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & & \vdots \\ 0 & \cdots & 0 & x_c & -x_{c-1} & 0 & \cdots & 0 \end{pmatrix},$$

whose rows are global generators of  $\mathcal{A} \cap \mathcal{O}^c \times \{0\} \times \cdots \times \{0\}$ . Consider the matrix  $\begin{pmatrix} \gamma_{1,1} & \cdots & \gamma_{1,k'} \\ \vdots & & \vdots \\ \gamma_{(c-1)!,1} & \cdots & \gamma_{(c-1)!,k'} \end{pmatrix} \begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_{k'} \end{pmatrix}$ . Its  $(i, j)$  components,  $i = 1, \dots, (c-1)!$ ,  $j = (c-1)! + 1, \dots, k'$ , are close to 0. Hence by (\*) there exist  $C^\omega$  functions  $\delta_{i,j}$ ,  $i = 1, \dots, (c-1)!$ ,  $j = 1, \dots, k'$ , close to 0 such that the  $(i, j)$  components,  $i = 1, \dots, (c-1)!$ ,  $j = (c-1)! + 1, \dots, k'$ , of the matrix

$$\begin{pmatrix} \gamma_{1,1} & \cdots & \gamma_{1,k'} \\ \vdots & & \vdots \\ \gamma_{(c-1)!,1} & \cdots & \gamma_{(c-1)!,k'} \end{pmatrix} \begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_{k'} \end{pmatrix} - \begin{pmatrix} \delta_{1,1} & \cdots & \delta_{1,k'} \\ \vdots & & \vdots \\ \delta_{(c-1)!,1} & \cdots & \delta_{(c-1)!,k'} \end{pmatrix} \begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_{k'} \end{pmatrix}$$

are 0. Replace  $\gamma_{i,j}$  with  $\tilde{\gamma}_{i,j} = \gamma_{i,j} - \delta_{i,j}$ . Then the  $(i, j)$  components,  $i = 1, \dots, (c-1)!$ ,  $j = (c-1)! + 1, \dots, k'$ , of the matrix  $\begin{pmatrix} \tilde{\gamma}_{1,1} & \cdots & \tilde{\gamma}_{1,k'} \\ \vdots & & \vdots \\ \tilde{\gamma}_{(c-1)!,1} & \cdots & \tilde{\gamma}_{(c-1)!,k'} \end{pmatrix} \begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_{k'} \end{pmatrix}$  are 0, and each row is an

approximation of the corresponding row of the matrix  $\begin{pmatrix} x_2 & -x_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & x_c & -x_{c-1} & 0 & \cdots & 0 \end{pmatrix}$ .

Therefore, we can suppose from the beginning that  $k = c - 1$ ,  $k' = (c - 1)!$ ,  $g_1 = (x_2, x_1, 0, \dots, 0), \dots, g_{k'} = (0, \dots, 0, x_c, -x_{c-1})$ .

Let  $\mathfrak{m}_x$  denote the maximal ideal of  $\mathcal{O}_x$  for  $x \in U$ . For each  $x \in C$ ,  $\tilde{g}_{1x}, \dots, \tilde{g}_{k'x}$  generate  $\mathcal{A}_x$  if and only if  $\tilde{g}_{1x}, \dots, \tilde{g}_{k'x}$  and  $\mathfrak{m}_x \mathcal{A}_x$  generate  $\mathcal{A}_x$  by Nakayama lemma. On the other hand, the images of  $g_{1x}, \dots, g_{k'x}$  in the linear space  $\mathcal{A}_x / \mathfrak{m}_x \mathcal{A}_x$ ,  $x \in C$ , are a basis and hence  $\mathcal{A}_x / \mathfrak{m}_x \mathcal{A}_x$  is a linear space of dimension  $k'$ . Hence it suffices to see that the images of  $\tilde{g}_{1x}, \dots, \tilde{g}_{k'x}$  in  $\mathcal{A}_x / \mathfrak{m}_x \mathcal{A}_x$ ,  $x \in C$ , are linearly independent. Here  $\mathfrak{m}_x \mathcal{A}_x = \mathfrak{m}_x^2 \mathcal{O}_x^c \cap \mathcal{A}_x$  because clearly  $\mathfrak{m}_x \mathcal{A}_x \subset \mathfrak{m}_x^2 \mathcal{O}_x^c \cap \mathcal{A}_x$  and  $\mathcal{A}_x / (\mathfrak{m}_x^2 \cap \mathcal{A}_x) (\approx (\mathfrak{m}_x^2 \mathcal{O}_x^c + \mathcal{A}_x) / \mathfrak{m}_x^2 \mathcal{O}_x^c)$  and  $\mathcal{A}_x / \mathfrak{m}_x \mathcal{A}_x$  are linear spaces of the same dimension. Now  $\cup_{x \in U} \mathcal{O}_x^c / \mathfrak{m}_x^2 \mathcal{O}_x^c$  coincides with the space of 1-jets from  $U$  to  $\mathbf{R}^c$ . Hence for  $x \in C$ , the images of  $\tilde{g}_{1x}, \dots, \tilde{g}_{k'x}$  in  $\mathcal{O}_x^c / \mathfrak{m}_x^2 \mathcal{O}_x^c$ , i.e. in  $\mathcal{A}_x / (\mathfrak{m}_x^2 \mathcal{O}_x^c \cap \mathcal{A}_x)$  are linearly independent because  $\tilde{g}_1, \dots, \tilde{g}_{k'}$  are sufficiently close to  $g_1, \dots, g_{k'}$ , respectively, in the Whitney  $C^1$  topology and the images of  $g_{1x}, \dots, g_{k'x}$  are linearly independent. □

### Remark 2.3.

- (1) In lemma 2.2,  $\tau|_C$  is an embedding of  $C$  into  $U$  close to id in the strong Whitney  $C^\infty$  topology such that  $\tau(C) = \tilde{C}$ . Conversely, assume that there exist an analytic embedding  $\tau_C$  of  $C$  into  $U$  close to id in the same topology. Then  $\tau_C$  is extensible to an analytic diffeomorphism  $\tau$  of  $U$  close to id in the topology. Define  $\tilde{C}$ ,  $\tilde{h}_i$ ,  $\tilde{g}_i$  and  $\tilde{\pi} : \tilde{M} \rightarrow \tilde{U}$  to be  $\tau(C)$ ,  $\tilde{h}_i \circ \tau^{-1}$ ,  $\tilde{g}_i \circ \tau^{-1}$  and the blowing-up of  $U$  along center  $\tilde{C}$ , respectively. Realize  $M$  and  $\tilde{M}$  in  $U \times \mathbf{P}(k)$  as before. Then  $\tilde{h}_i$  and  $\tilde{g}_i$  are close to  $h_i$  and  $g_i$  respectively,  $\sum_{j=0}^k \tilde{g}_{i,j} \tilde{h}_j = 0$ , and hence by lemma 2.2 there exists an analytic diffeomorphism  $\psi : M \rightarrow \tilde{M}$  close to id in the topology such that  $\tilde{\pi} \circ \psi = \tau \circ \pi$ .

When there exists this kind of  $\tau_C$ , we say  $\tilde{C}$  is *close* to  $C$  in the strong Whitney  $C^\infty$  topology. Let  $\psi : M_1 \rightarrow M_2$  and  $\tilde{\psi} : \tilde{M}_1 \rightarrow \tilde{M}_2$  be analytic maps between analytic manifolds with  $M_1 \subset \mathbf{R}^{n_1}$ ,  $\tilde{M}_1 \subset \mathbf{R}^{n_1}$ ,  $M_2 \subset \mathbf{R}^{n_2}$  and  $\tilde{M}_2 \subset \mathbf{R}^{n_2}$ . Assume that  $\tilde{M}_1$  is close to  $M_1$  in the topology through an analytic diffeomorphism  $\tau : M_1 \rightarrow \tilde{M}_1$  close to id in the topology. Then we say  $\tilde{\psi}$  is *close* to  $\psi$  in the topology if  $\tilde{\psi} \circ \tau$  is so to  $\psi$ .

- (2) The germ case of lemma 2.2 holds in the following sense. Let  $h_i, g_i, U$  and  $C$  be the same as above. Let  $X$  be a compact subset of  $U$ , and let  $\tilde{h}_i$  and  $\tilde{g}_i$  be  $C^\omega$  functions and maps defined on an open neighborhood  $V$  of  $X$  in  $U$  close to  $h_i|_V$  and  $g_i|_V$ , respectively, in the compact-open  $C^\infty$  topology with  $\sum_{j=0}^k \tilde{g}_{i,j} \tilde{h}_j = 0$ . Shrink  $V$ . Then the same statement as the former half of lemma 2.2 holds. For the latter half, let  $\pi : M \rightarrow U$  and  $\tilde{\pi} : \tilde{M} \rightarrow V$  denote the blowings-up along centers  $C$  and  $\tilde{C} = Z(\tilde{h}_i)$ . Let  $M \subset U \times \mathbf{P}(k)$  and  $\tilde{M} \subset V \times \mathbf{P}(k)$  be as above. Then there exist analytic embeddings  $\tau_- : V \rightarrow U$  and  $\psi_- : \tilde{M} \rightarrow M$  close to  $\text{id}$  in the compact-open  $C^\infty$  topology such that  $\tau_- \circ \tilde{\pi} = \pi \circ \psi_-$ . In this case we say  $\tilde{C}$  is *close to  $C$  at  $X$*  in the  $C^\infty$  topology, and define *closeness* of an analytic map to another one at a compact set.

**2.4. Nash approximation.** We state and prove a Nash approximation theorem of an analytic desingularization of a Nash function. This result will be crucial for the proof of theorem 1.11.

**Theorem 2.4.** *Let  $f$  be a Nash function on a Nash manifold  $M$ . Let  $M_m \xrightarrow{\pi_m} M_{m-1} \rightarrow \dots \xrightarrow{\pi_1} M_0 = M$  be a sequence of blowings-up along smooth analytic centers  $C_{m-1}$  in  $M_{m-1}, \dots, C_0$  in  $M_0$ , respectively, such that  $f \circ \pi_1 \circ \dots \circ \pi_m$  has only normal crossing singularities. Let  $X$  be a compact subset of  $M$ . Then there exist an open semialgebraic neighborhood  $U$  of  $X$  in  $M$ , a sequence of blowings-up  $U_m \xrightarrow{\tau_m} U_{m-1} \rightarrow \dots \xrightarrow{\tau_1} U_0 = U$  along smooth Nash centers  $D_{m-1}$  in  $U_{m-1}, \dots, D_0$  in  $U_0$ , respectively, and an analytic embedding  $\psi : U_m \rightarrow M_m$  such that  $\psi(\tau_m^{-1}(D_{m-1})) \subset \pi_m^{-1}(C_{m-1}), \dots, \psi((\tau_1 \circ \dots \circ \tau_m)^{-1}(D_0)) \subset (\pi_1 \circ \dots \circ \pi_m)^{-1}(C_0)$  and  $f \circ \pi_1 \circ \dots \circ \pi_m \circ \psi = f \circ \tau_1 \circ \dots \circ \tau_m$ . Let  $M_1, \dots, M_m$  be realized in  $M \times \mathbf{P}(k_0), \dots, M \times \mathbf{P}(k_0) \times \dots \times \mathbf{P}(k_{m-1})$ , respectively, for some  $k_0, \dots, k_{m-1} \in \mathbf{N}$ . Then we can realize  $U_1, \dots, U_m$  in  $U \times \mathbf{P}(k_0), \dots, U \times \mathbf{P}(k_0) \times \dots \times \mathbf{P}(k_{m-1})$ , respectively, so that each pair  $D_i \subset U_i$  and  $\psi$  are close to  $C_i \subset M_i$  at  $(\tau_1 \circ \dots \circ \tau_{i-1})^{-1}(X)$  and to  $\text{id}$  at  $(\tau_1 \circ \dots \circ \tau_m)^{-1}(X)$ , respectively, in the  $C^\infty$  topology.*

The proof of theorem 2.4 is the heart of the paper. It consists in a combination of algebra and topology, via a nested Néron Desingularization Theorem (see Theorem 11.4, [21]) and Nash Approximation Theorem. We proceed as follows. First we describe the analytic situation of the sequence of blowings-up in terms of ideals. Next we apply the nested version of Néron Desingularization Theorem and come down to a regular situation. Then, in the regular situation, the classical Nash Approximation Theorem enables to realize the approximation. The idea comes from the proof of Theorem 1.1 in [3], where the usual Néron Desingularization Theorem is used.

*Proof.* Consider the blowing-up  $\pi_1 : M_1 \rightarrow M_0$  along center  $C_0$ . Let

- $\mathcal{I}^0$  denote the sheaf of  $\mathcal{O}$ -ideals on  $M_0$  defined by  $C_0$ ,
- $h_0^0, \dots, h_{k_0}^0$  its global generators,
- $\mathcal{A}^0 \subset \mathcal{O}_{M_0}^{k_0+1}$  the sheaf of relations of  $h_0^0, \dots, h_{k_0}^0$ ,
- $g_1^0 = (g_{1,0}^0, \dots, g_{1,k_0}^0), \dots, g_{k'_0}^0 = (g_{k'_0,0}^0, \dots, g_{k'_0,k_0}^0)$  global generators of  $\mathcal{A}^0$ ,
- $\xi_1^0, \dots, \xi_{s_0}^0$  generators of the ideal  $I(\mathbf{P}(k_0))$  of  $\mathbf{R}[y_{i,j}^0]_{0 \leq i,j \leq k_0}$  of functions vanishing on  $\mathbf{P}(k_0)$  in  $\mathbf{R}^{(k_0+1)^2}$ .

Set  $l_{i_1, i_2, i_3}^0(x, y^0) = y_{i_1, i_2}^0 h_{i_3}^0(x) - y_{i_3, i_1}^0 h_{i_2}^0(x)$  for  $(x, y^0) \in M_0 \times \mathbf{R}^{(k_0+1)^2}$  and for  $i_1, i_2, i_3 = 0, \dots, k_0$ . Then

$$(1) \quad \sum_{j=0}^{k_0} g_{i,j}^0(x) h_j^0(x) = 0 \quad \text{on } M_0 \text{ for } i = 1, \dots, k'_0,$$

and  $M_1$  is generated by those  $l_{i_1, i_2, i_3}^0, \sum_{j=0}^{k_0} y_{j, i_1}^0 g_{i_2, j}^0$  and  $\xi_i^0$ , namely

$$M_1 = \{(x, y^0) \in M_0 \times \mathbf{R}^{(k_0+1)^2} : l_{i_1, i_2, i_3}^0(x, y^0) = 0, i_1, i_2, i_3 = 0, \dots, k_0,$$

$$\sum_{j=0}^{k_0} y_{j, i_1}^0 g_{i_2, j}^0(x) = 0, i_1 = 0, \dots, k_0, i_2 = 1, \dots, k'_0, \text{ and } \xi_i^0(y^0) = 0, i = 1, \dots, s_0\}.$$

Let  $\{\alpha_i^1\}$  denote the generators. Note that  $\pi_1$  is the restriction to  $M_1$  of the projection  $M_0 \times \mathbf{R}^{(k_0+1)^2} \rightarrow M_0$ . According to definition 1.1, we assume that  $C_0 \subset \text{Sing } f$ , which is described as follows. Let  $v_1, \dots, v_n$  be Nash vector fields on  $M_0$  which span the tangent space of  $M_0$  at each point of  $M_0$ . Then we see, as previously, that  $C_0 \subset \text{Sing } f$  if and only if there exist  $C^\omega$  functions  $a_{i, j}^0$  on  $M_0$ ,  $i = 1, \dots, n$ ,  $j = 0, \dots, k_0$ , such that

$$(2) \quad v_i f = \sum_{j=0}^{k_0} a_{i, j}^0 h_j^0 \quad \text{on } M_0 \text{ for } i = 1, \dots, n.$$

Let  $\tilde{h}_0^0, \dots, \tilde{h}_{k_0}^0, \tilde{g}_1^0, \dots, \tilde{g}_{k'_0}^0, \tilde{a}_{i, j}^0$ ,  $i = 1, \dots, n$ ,  $j = 0, \dots, k_0$ , be  $C^\omega$  approximations of  $h_0^0, \dots, h_{k_0}^0, g_1^0, \dots, g_{k'_0}^0, a_{i, j}^0$ ,  $i = 1, \dots, n$ ,  $j = 0, \dots, k_0$ , respectively, in the strong Whitney  $C^\infty$  topology such that

$$\begin{aligned} (\tilde{1}) \quad & \sum_{j=0}^{k_0} \tilde{g}_{i, j}^0 \tilde{h}_j^0 = 0 \text{ for } i = 1, \dots, k'_0 \text{ and} \\ (\tilde{2}) \quad & v_i f = \sum_{j=0}^{k_0} \tilde{a}_{i, j}^0 \tilde{h}_j^0 \text{ for } i = 1, \dots, n. \end{aligned}$$

Then by lemma 2.2, the common zero set  $\tilde{C}_0 = Z(\tilde{h}_j^0)$  is smooth and of the same dimension as  $C_0$ ,  $\tilde{g}_1^0, \dots, \tilde{g}_{k'_0}^0$  are generators of the sheaf of relations  $\tilde{\mathcal{A}}^0$  of  $\tilde{h}_0^0, \dots, \tilde{h}_{k_0}^0$ , and if we let  $M_0 \times \mathbf{R}^{(k_0+1)^2} \supset M_0 \times \mathbf{P}(k_0) \supset \tilde{M}_1 \xrightarrow{\tilde{\pi}_1} M_0$  denote the blowing-up of  $M_0$  along center  $\tilde{C}_0$  defined by  $\tilde{h}_0^0, \dots, \tilde{h}_{k_0}^0$  then there exist analytic diffeomorphisms  $\psi_0$  of  $M_0$  and  $\tilde{\psi}_0 : M_1 \rightarrow \tilde{M}_1$  close to id in the strong Whitney  $C^\infty$  topology such that  $\psi_0(C_0) = \tilde{C}_0$  and  $\tilde{\pi}_1 \circ \tilde{\psi}_0 = \psi_0 \circ \pi_1$ . Hence  $f \circ \tilde{\pi}_1 : \tilde{M}_1 \rightarrow \mathbf{R}$  is close to  $f \circ \pi_1 : M_1 \rightarrow \mathbf{R}$  in the strong Whitney  $C^\infty$  topology because if we regard  $f$  as a function on  $M_0 \times \mathbf{R}^{(k_0+1)^2}$  then  $f \circ \tilde{\pi}_1 = f|_{\tilde{M}_1}$  and  $f \circ \pi_1 = f|_{M_1}$ . Moreover,  $\tilde{C}_0 \subset \text{Sing } f$  by  $(\tilde{2})$  for  $\tilde{h}_j^0$ , and  $I(\tilde{M}_1)$  is generated by  $\tilde{l}_{i_1, i_2, i_3}^0(x, y^0)$  which is defined by  $\tilde{l}_{i_1, i_2, i_3}^0(x, y^0) = y_{i_1, i_2}^0 \tilde{h}_{i_3}^0(x) - y_{i_3, i_1}^0 \tilde{h}_{i_2}^0(x)$ ,  $\sum_{j=0}^{k_0} y_{j, i_1}^0 \tilde{g}_{i_2, j}^0(x)$  and  $\xi_i^0(y^0)$  in  $C^\omega(M_0 \times \mathbf{R}^{(k_0+1)^2})$ . Let  $\tilde{\alpha}_i^1$  denote the generators corresponding to  $\alpha_i^1$ .

Consider the second blowing-up  $\pi_2 : M_2 \rightarrow M_1$  along  $C_1$ . In the same way as for the first blowing-up we define

- $\mathcal{I}^1 \subset \mathcal{O}_{M_1}$ ,
- $h_0^1, \dots, h_{k_1}^1 \in H^0(M_1, \mathcal{I}^1)$ ,
- $\mathcal{A}^1 \subset \mathcal{O}_{M_1}^{k_1+1}$ ,
- $g_1^1 = (g_{1,0}^1, \dots, g_{1, k_1}^1), \dots, g_{k_1}^1 = (g_{k_1,0}^1, \dots, g_{k_1, k_1}^1) \in H^0(M_1, \mathcal{A}^1)$ ,
- $\xi_{s_1}^1, \dots, \xi_{s_1}^1 \in \mathbf{R}[y_{i, j}^1]_{0 \leq i, j \leq k_1}$ ,
- $l_{i_1, i_2, i_3}^1(x, y^0, y^1) = y_{i_1, i_2}^1 h_{i_3}^1(x, y^0) - y_{i_3, i_1}^1 h_{i_2}^1(x, y^0)$  for  $(x, y^0, y^1) \in M_1 \times \mathbf{R}^{(k_1+1)^2}$  and for  $i_1, i_2, i_3 = 0, \dots, k_1$ ,
- $C^\omega$  functions  $a_{i, j}^1$  on  $M_1$  for  $i = 1, \dots, n$ ,  $j = 0, \dots, k_1$

so that

$$(1) \quad \sum_{j=0}^{k_1} g_{i, j}^1(x, y^0) h_j^1(x, y^0) = 0 \quad \text{on } M_1 \text{ for } i = 1, \dots, k'_1,$$

$$(2) \quad v_i f(x) = \sum_{j=0}^{k_1} a_{i,j}^1(x, y^0) h_j^1(x, y^0) \quad \text{on } M_1 \text{ for } i = 1, \dots, n,$$

$$M_2 = \{(x, y^0, y^1) \in M_1 \times \mathbf{R}^{(k_1+1)^2} : l_{i_1, i_2, i_3}^1(x, y^0, y^1) = 0, i_1, i_2, i_3 = 0, \dots, k_1,$$

$$\sum_{j=0}^{k_1} y_{j, i_1}^1 g_{i_2, j}^1(x, y^0) = 0, i_1 = 0, \dots, k_1, i_2 = 1, \dots, k'_1, \text{ and } \xi_i^1(y^1) = 0, i = 1, \dots, s_1\},$$

where  $I(M_2)$  in  $C^\omega(M_1 \times \mathbf{R}^{(k_1+1)^2})$  is generated by those functions, denoted by  $\{\alpha_i^2\}$ , in the last braces, and  $\pi_2$  is the restriction to  $M_2$  of the projection  $M_1 \times \mathbf{R}^{(k_1+1)^2} \rightarrow M_1$ .

Here we require as another prescription of blowings-up that  $C_1$  is normal crossing with  $\pi_1^{-1}(C_0)$ . For each  $(x, y^0) \in C_1 \cap \pi_1^{-1}(C_0)$  there are two possible cases to consider:  $C_1$  is transversal to  $\pi_1^{-1}(C_0)$  at  $(x, y^0)$  or  $C_1(x, y^0) \subset \pi_1^{-1}(C_0)_{(x, y^0)}$ . Divide, if necessary,  $C_1$  into two unions  $C_1^1$  and  $C_1^2$  of its connected components so that on each union, only one case arises, and regard  $\pi_1 : M_1 \rightarrow M_0$  as a composition  $\pi_1 = \pi_1^2 \circ \pi_1^1$ , where  $\pi_1^1$  is the blowing-up along center  $C_1^1$  and  $\pi_1^2$  is the blowing-up along center  $\pi_1^{1-1}(C_1^2)$ . Then we can assume that globally  $C_1$  is transversal to  $\pi_1^{-1}(C_0)$  or  $C_1 \subset \pi_1^{-1}(C_0)$ . The latter case occurs if and only if there exist  $C^\omega$  functions  $b_{j_0, j_1}^1$  on  $M_1$ ,  $j_0 = 0, \dots, k_0$ ,  $j_1 = 0, \dots, k_1$ , such that

$$(3) \quad h_{j_0}^0(x) = \sum_{j_1=0}^{k_1} b_{j_0, j_1}^1(x, y^0) h_{j_1}^1(x, y^0) \quad \text{on } M_1 \text{ for } j_0 = 0, \dots, k_0.$$

We extend  $h_j^1, g_{i,j}^1, a_{i,j}^1$  and  $b_{j_0, j_1}^1$  (if they exist) to analytic functions on  $M_0 \times \mathbf{R}^{(k_0+1)^2}$ . For simplicity we use the same notations for the extensions. Then (1), (2) and (3) become

$$(1) \quad \sum_j g_{i,j}^1 h_j^1 = \sum_{i_1} \beta_{1, i_1, i}^1 \alpha_{i_1}^1 \quad \text{on } M_0 \times \mathbf{R}^{(k_0+1)^2},$$

$$(2) \quad v_i f = \sum_j a_{i,j}^1 h_j^1 + \sum_{i_2} \beta_{2, i_2, i}^1 \alpha_{i_2}^1 \quad \text{on } M_0 \times \mathbf{R}^{(k_0+1)^2},$$

$$(3) \quad h_{j_0}^0 = \sum_{j_1} b_{j_0, j_1}^1 h_{j_1}^1 + \sum_i \beta_{3, i, j_0}^1 \alpha_i^1 \quad \text{on } M_0 \times \mathbf{R}^{(k_0+1)^2}$$

for some  $C^\omega$  functions  $\beta_{i_1, i_2, i_3}^i$  on  $M_0 \times \mathbf{R}^{(k_0+1)^2}$ . Now  $Z(h_j^1) \cap M_1 = C_1$ , and  $I(M_2)$  in  $C^\omega(M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2})$  is generated by  $\alpha_i^1$  and  $\alpha_i^2$ , which we naturally regard as  $C^\omega$  functions on  $M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2}$ .

For the second blowing-up, we consider again  $C^\omega$  approximations  $\tilde{h}_0^1, \dots, \tilde{h}_{k_1}^1, \tilde{g}_1^1 = (\tilde{g}_{1,0}^1, \dots, \tilde{g}_{1, k_1}^1), \dots, \tilde{g}_{k'_1}^1 = (\tilde{g}_{k'_1, 0}^1, \dots, \tilde{g}_{k'_1, k_1}^1), \tilde{a}_{i,j}^1, i = 1, \dots, n, j = 0, \dots, k_1, \tilde{b}_{j_0, j_1}^1, j_0 = 0, \dots, k_0, j_1 = 0, \dots, k_1$  and  $\tilde{\beta}_{i_1, i_2, i_3}^1$  of  $h_0^1, \dots, h_{k_1}^1, g_1^1 = (g_{1,0}^1, \dots, g_{1, k_1}^1), \dots, g_{k'_1}^1 = (g_{k'_1, 0}^1, \dots, g_{k'_1, k_1}^1), a_{i,j}^1, b_{j_0, j_1}^1$  (if exist) and  $\beta_{i_1, i_2, i_3}^1$  on  $M_0 \times \mathbf{R}^{(k_0+1)^2}$  in the strong Whitney  $C^\infty$  topology such that

$$(\tilde{1}) \quad \sum_j \tilde{g}_{i,j}^1 \tilde{h}_j^1 = \sum_{i_1} \tilde{\beta}_{1, i_1, i}^1 \tilde{\alpha}_{i_1}^1 \quad \text{on } M_0 \times \mathbf{R}^{(k_0+1)^2} \text{ for } i = 1, \dots, k'_1,$$

$$(\tilde{2}) \quad v_i f = \sum_j \tilde{a}_{i,j}^1 \tilde{h}_j^1 + \sum_{i_2} \tilde{\beta}_{2, i_2, i}^1 \tilde{\alpha}_{i_2}^1 \quad \text{on } M_0 \times \mathbf{R}^{(k_0+1)^2} \text{ for } i = 1, \dots, n,$$

$$(\tilde{3}) \quad \tilde{h}_{j_0}^0 = \sum_{j_1} \tilde{b}_{j_0, j_1}^1 \tilde{h}_{j_1}^1 + \sum_i \tilde{\beta}_{3, i, j_0}^1 \tilde{\alpha}_i^1 \quad \text{on } M_0 \times \mathbf{R}^{(k_0+1)^2} \text{ for } j_0 = 0, \dots, k_0.$$

Then  $\tilde{C}_1 = Z(\tilde{h}_j^1) \cap \tilde{M}_1$  is smooth and of the same dimension as  $C_1$ . If  $C_1 \subset \pi_1^{-1}(C_0)$ , then  $\tilde{C}_1$  is contained in  $\tilde{\pi}_1^{-1}(\tilde{C}_0)$  by (3) and  $(\tilde{3})$ . If  $C_1 \not\subset \pi_1^{-1}(C_0)$ , i.e. if  $C_1$  is transversal to  $\pi_1^{-1}(C_0)$  in  $M_1$ ,  $\tilde{C}_1$  is transversal to  $\tilde{\pi}_1^{-1}(\tilde{C}_0)$  in  $\tilde{M}_1$  because the above diffeomorphism  $\tilde{\psi}_0 : M_1 \rightarrow \tilde{M}_1$  is close to id in the strong Whitney  $C^\infty$  topology and carries  $\pi_1^{-1}(C_0)$  to

$\tilde{\pi}_1^{-1}(\tilde{C}_0)$  and because  $\tilde{C}_1$  is close to  $C_1$  in the strong Whitney  $C^\infty$  topology. Hence, in any case  $\tilde{C}_1$  is normal crossing with  $\tilde{\pi}_1^{-1}(\tilde{C}_0)$ . It also follows from (2) that  $\tilde{C}_1 \subset \text{Sing } f = \tilde{\pi}_1^{-1}(\text{Sing } f)$ . Thus we can take the blowing-up  $\tilde{\pi}_2 : \tilde{M}_2 \rightarrow \tilde{M}_1$  of  $\tilde{M}_1$  along center  $\tilde{C}_1$ , and embed  $\tilde{M}_2$  by  $\tilde{h}_0^1, \dots, \tilde{h}_{k_1}^1$  into  $\tilde{M}_1 \times \mathbf{P}(k_1) \subset \tilde{M}_1 \times \mathbf{R}^{(k_1+1)^2} \subset M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2}$  so that  $\tilde{\pi}_2$  is the restriction to  $\tilde{M}_2$  of the projection  $\tilde{M}_1 \times \mathbf{R}^{(k_1+1)^2} \rightarrow \tilde{M}_1$ . Then there exist analytic diffeomorphisms  $\psi_1 : M_1 \rightarrow \tilde{M}_1$  and  $\tilde{\psi}_1 : M_2 \rightarrow \tilde{M}_2$  close to id in the strong Whitney  $C^\infty$  topology ( $\psi_1$  is not necessarily equal to  $\psi_0$ ) such that  $\psi_1(C_1) = \tilde{C}_1$  and  $\tilde{\pi}_2 \circ \tilde{\psi}_1 = \psi_1 \circ \pi_2$ ;  $f \circ \tilde{\pi}_1 \circ \tilde{\pi}_2 : \tilde{M}_2 \rightarrow \mathbf{R}$  is close to  $f \circ \pi_1 \circ \pi_2 : M_2 \rightarrow \mathbf{R}$  in the strong Whitney  $C^\infty$  topology;  $I(\tilde{M}_2)$  is generated in  $C^\omega(M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2})$  by  $\tilde{l}_{i_1, i_2, i_3}(x, y^0, y^1) = y_{i_1, i_2}^1 \tilde{h}_{i_3}^1(x, y^0) - y_{i_3, i_1}^1 \tilde{h}_{i_2}^1(x, y^0)$ ,  $\sum_{j=0}^{k_1} y_{j, i_1}^1 \tilde{g}_{i_2, j}^1(x, y^0)$ ,  $\xi_i^1(y^1)$  and  $\tilde{\alpha}_i^1$ . Let  $\tilde{\alpha}_i^2$  denote the former generators, and let  $\tilde{\alpha}_i^1$  be naturally extended to  $M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2}$ .

Note that there exists a  $C^\omega$  diffeomorphism from  $M_2$  to  $\tilde{M}_2$  close to id in the strong Whitney  $C^\infty$  topology and carrying  $\pi_2^{-1}(C_1) \cup (\pi_1 \circ \pi_2)^{-1}(C_0)$  to  $\tilde{\pi}_2^{-1}(\tilde{C}_1) \cup (\tilde{\pi}_1 \circ \tilde{\pi}_2)^{-1}(\tilde{C}_0)$  for the following reason. First by lemma 4.3 in [6], we have a  $C^\omega$  diffeomorphism from  $M_1$  to  $\tilde{M}_1$  close to id in the strong Whitney  $C^\infty$  topology and carrying  $C_1 \cup \pi_1^{-1}(C_0)$  to  $\tilde{C}_1 \cup \tilde{\pi}_1^{-1}(\tilde{C}_0)$ . Hence we can assume that  $C_1 \cup \pi_1^{-1}(C_0) = \tilde{C}_1 \cup \tilde{\pi}_1^{-1}(\tilde{C}_0)$ . Then in the same way as in the proof of lemma 2.2 we construct a  $C^\omega$  diffeomorphism  $\eta : M_2 \rightarrow \tilde{M}_2$  close to id in the strong Whitney  $C^\infty$  topology such that  $\tilde{\pi}_2 \circ \eta = \pi_2$  and hence  $\eta(\pi_2^{-1}(C_1) \cup (\pi_1 \circ \pi_2)^{-1}(C_0)) = \eta(\pi_2^{-1}(C_1 \cup \pi_1^{-1}(C_0))) = \tilde{\pi}_2^{-1}(C_1 \cup \pi_1^{-1}(C_0)) = \tilde{\pi}_2^{-1}(\tilde{C}_1) \cup (\tilde{\pi}_1 \circ \tilde{\pi}_2)^{-1}(\tilde{C}_0)$ .

We repeat the same arguments inductively on each blowing-up. Then condition (3) becomes somewhat complicated because the union of the inverse images of the previous centers is not necessarily smooth. Let us consider the center  $C_2$  of the blowing-up  $\pi_3 : M_3 \rightarrow M_2$ . We describe the condition that  $C_2$  is normal crossing with  $A = \pi_2^{-1}(C_1) \cup (\pi_1 \circ \pi_2)^{-1}(C_0)$  as follows. Let  $h_j^2, g_j^2 = (g_{j,0}^2, \dots, g_{j,k_2}^2), \xi_i^2, a_{i,j}^2, \beta_{i_1, i_2, i_3}^2$  on  $M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2}$  and their  $C^\omega$  approximations  $\tilde{h}_j^2, \tilde{g}_j^2 = (\tilde{g}_{j,0}^2, \dots, \tilde{g}_{j,k_2}^2), \tilde{\xi}_i^2, \tilde{a}_{i,j}^2, \tilde{\beta}_{i_1, i_2, i_3}^2$  in the strong Whitney  $C^\infty$  topology be given for the center  $C_2$  so that the corresponding equalities to (1), (1'), (2) and (2') hold. Set  $Y = \overline{A - \pi_2^{-1}(C_1)}$ . Then  $Y$  is a smooth analytic set of codimension 1 in  $M_2$ . If  $C_1 \not\subset \pi_1^{-1}(C_0)$  then  $Y = A$ ; if  $C_1 \subset \pi_1^{-1}(C_0)$  then  $\pi_2^{-1}(C_1) \subset (\pi_1 \circ \pi_2)^{-1}(C_0)$  and  $Y \cup \pi_2^{-1}(C_1)$  is a decomposition of  $(\pi_1 \circ \pi_2)^{-1}(C_0)$  to smooth analytic sets, it follows from the normal crossing property of  $C_2$  with  $A$  that  $C_2$  is normal crossing with  $\pi_2^{-1}(C_1)$  and with  $Y$  (the converse is not necessarily correct), and  $I(Y)$  in  $C^\omega(M_2)$  coincides with

$$\{g \in C^\omega(M_2) : gh_i^1 = \sum_{j=0}^{k_0} c_{i,j} h_j^0 \text{ for some } c_{i,j} \in C^\omega(M_2), i = 0, \dots, k_1\}.$$

Let  $\chi_j^{0,2}, j = 1, \dots, t_2$ , be generators of  $I(Y)$ . Then there exist  $C^\omega$  functions  $c_{j_0, j_1, j}^{0,2}$  on  $M_2, j_0 = 0, \dots, k_0, j_1 = 1, \dots, k_1, j = 1, \dots, t_2$ , such that

$$(4) \quad \chi_j^{0,2}(x, y^0, y^1) h_{j_1}^1(x, y^0) = \sum_{j_0=0}^{k_0} c_{j_0, j_1, j}^{0,2}(x, y^0, y^1) h_{j_0}^0(x) \quad \text{on } M_2,$$

and as in the case of the second blowing-up, dividing  $C_2$  if necessary we can assume that  $C_2$  is transversal to  $\pi_2^{-1}(C_1)$  or contained in  $\pi_2^{-1}(C_1)$  and that  $C_2$  is transversal to  $Y$  or contained in  $Y$ . If  $C_2 \subset \pi_2^{-1}(C_1)$  then there exist  $C^\omega$  functions  $b_{j_1, j_2}^2$  on  $M_2, j_1 = 0, \dots, k_1, j_2 = 0, \dots, k_2$ , such that

$$(3) \quad h_{j_1}^1(x, y^0) = \sum_{j_2=0}^{k_2} b_{j_1, j_2}^2(x, y^0, y^1) h_{j_2}^2(x, y^0, y^1) \quad \text{on } M_2,$$

and if  $C_2 \subset Y$  then there exist  $C^\omega$  functions  $d_{j,j_2}^{0,2}$  on  $M_2$ ,  $j = 1, \dots, t_2$ ,  $j_2 = 0, \dots, k_2$ , such that

$$(5) \quad \chi_j^{0,2}(x, y^0, y^1) = \sum_{j_2=0}^{k_2} d_{j,j_2}^{0,2}(x, y^0, y^1) h_{j_2}^2(x, y^0, y^1) \quad \text{on } M_2.$$

As before we assume that  $\chi_j^{0,2}$ ,  $b_{j_1,j_2}^2$ ,  $c_{j_1,j_2}^{0,2}$ ,  $d_{j,j_2}^{0,2}$  are defined on  $M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2}$ . Then there exist  $C^\omega$  functions  $\gamma_{i_1,i_2,i_3}^i$  and  $\gamma_{i_1,i_2,i_3,i_4}^i$  on  $M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2}$  such that

$$(3) \quad h_{j_1}^1 = \sum_{j_2} b_{j_1,j_2}^2 h_{j_2}^2 + \sum_i \gamma_{1,i,j_1}^1 \alpha_i^1 + \sum_i \gamma_{1,i,j_1}^2 \alpha_i^2,$$

$$(4) \quad \chi_j^{0,2} h_{j_1}^1 = \sum_{j_0} c_{j_0,j_1,j}^{0,2} h_{j_0}^0 + \sum_i \gamma_{2,i,j,j_1}^1 \alpha_i^1 + \sum_i \gamma_{2,i,j,j_1}^2 \alpha_i^2,$$

$$(5) \quad \chi_j^{0,2} = \sum_{j_2} d_{j,j_2}^{0,2} h_{j_2}^2 + \sum_i \gamma_{3,i,j}^1 \alpha_i^1 + \sum_i \gamma_{3,i,j}^2 \alpha_i^2 \quad \text{on } M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2}.$$

We need to consider also  $C^\omega$  approximations  $\tilde{\chi}_j^{0,2}$ ,  $\tilde{b}_{j_1,j_2}^2$ ,  $\tilde{c}_{j_1,j_2,j}^{0,2}$ ,  $\tilde{d}_{j,j_2}^{0,2}$ ,  $\tilde{\gamma}_{i_1,i_2,i_3}^i$  and  $\tilde{\gamma}_{i_1,i_2,i_3,i_4}^i$  of  $\chi_j^{0,2}$ ,  $b_{j_1,j_2}^2$  (if they exist),  $c_{j_1,j_2,j}^{0,2}$ ,  $d_{j,j_2}^{0,2}$  (if exist),  $\gamma_{i_1,i_2,i_3}^i$  (if exist) and  $\gamma_{i_1,i_2,i_3,i_4}^i$  on  $M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2}$  in the strong Whitney  $C^\infty$  topology such that

$$(3) \quad \tilde{h}_{j_1}^1 = \sum_{j_2=0}^{k_2} \tilde{b}_{j_1,j_2}^2 \tilde{h}_{j_2}^2 + \sum_i \tilde{\gamma}_{1,i,j_1}^1 \tilde{\alpha}_i^1 + \sum_i \tilde{\gamma}_{1,i,j_1}^2 \tilde{\alpha}_i^2,$$

$$(4) \quad \tilde{\chi}_j^{0,2} \tilde{h}_{j_1}^1 = \sum_{j_0} \tilde{c}_{j_0,j_1,j}^{0,2} \tilde{h}_{j_0}^0 + \sum_i \tilde{\gamma}_{2,i,j,j_1}^1 \tilde{\alpha}_i^1 + \sum_i \tilde{\gamma}_{2,i,j,j_1}^2 \tilde{\alpha}_i^2,$$

$$(5) \quad \tilde{\chi}_j^{0,2} = \sum_{j_2} \tilde{d}_{j,j_2}^{0,2} \tilde{h}_{j_2}^2 + \sum_i \tilde{\gamma}_{3,i,j}^1 \tilde{\alpha}_i^1 + \sum_i \tilde{\gamma}_{3,i,j}^2 \tilde{\alpha}_i^2 \quad \text{on } M_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2}.$$

Set  $\tilde{Y} = Z(\tilde{\chi}_j) \cap \tilde{M}_2$ . Then

$$\tilde{Y} = \overline{(\tilde{\pi}_1 \circ \tilde{\pi}_2)^{-1}(\tilde{C}_0) - \tilde{\pi}_2^{-1}(\tilde{C}_1)}$$

because  $\tilde{Y}$  contains the right hand side by (4) and because the opposite inclusion follows from the facts that  $Y$  and the right hand side are smooth and of codimension 1 in  $M_2$  and in  $\tilde{M}_2$ , respectively, and that  $\chi_j$  are generators of  $I(Y)$  in  $C^\omega(M_2)$ . Hence  $\tilde{\pi}_2^{-1}(\tilde{C}_1) \cup (\tilde{\pi}_1 \circ \tilde{\pi}_2)^{-1}(\tilde{C}_0)$ , which is normal crossing, is the union of the smooth analytic sets  $\tilde{\pi}_2^{-1}(\tilde{C}_1)$  and  $\tilde{Y}$ . Moreover,  $\tilde{C}_2$  is normal crossing with  $\tilde{\pi}_2^{-1}(\tilde{C}_1) \cup (\tilde{\pi}_1 \circ \tilde{\pi}_2)^{-1}(\tilde{C}_0)$  for the following four reasons.

If  $C_2$  is transversal to  $\pi_2^{-1}(C_1)$  or to  $Y$ , so is  $\tilde{C}_2$  to  $\tilde{\pi}_2^{-1}(\tilde{C}_1)$  or to  $\tilde{Y}$ , respectively, by the same reason as before. If  $C_2 \subset \pi_2^{-1}(C_1)$ , then there exist  $C^\omega$  functions  $b_{j_1,j_2}^2$  with (3) on  $M_2$ , hence  $\tilde{h}_{j_1}^1 = \sum_{j_2=0}^{k_2} \tilde{b}_{j_1,j_2}^2 \tilde{h}_{j_2}^2$  on  $\tilde{M}_2$  and  $\tilde{C}_2 \subset \tilde{\pi}_2^{-1}(\tilde{C}_1)$ . In the same way we see that if  $C_2 \subset Y$  then  $\tilde{C}_2 \subset \tilde{Y}$ . The fourth consideration is that  $C_2$  is normal crossing with  $\pi_2^{-1}(C_1) \cup (\pi_1 \circ \pi_2)^{-1}(C_0)$ .

By these four properties we can find also a  $C^\omega$  diffeomorphism from  $M_2$  to  $\tilde{M}_2$  close to id in the strong Whitney  $C^\infty$  topology and carrying  $C_2$ ,  $\pi_2^{-1}(C_1)$  and  $(\pi_1 \circ \pi_2)^{-1}(C_0)$  to  $\tilde{C}_2$ ,  $\tilde{\pi}_2^{-1}(\tilde{C}_1)$  and  $(\tilde{\pi}_1 \circ \tilde{\pi}_2)^{-1}(\tilde{C}_0)$ , respectively.

Let  $1 < m' (< m) \in \mathbf{N}$ . As above we inductively embed  $M_{m'}$  into  $M_{m'-1} \times \mathbf{R}^{(k_{m'-1}+1)^2} (\subset M_0 \times \mathbf{R}^{(k_0+1)^2} \times \dots \times \mathbf{R}^{(k_{m'-1}+1)^2})$  and obtain a finite number of  $C^\omega$  functions on  $M_0 \times \dots \times \mathbf{R}^{(k_{m'-1}+1)^2}$ , namely  $h_j^{m'}$ ,  $a_{i,j}^{m'}$ ,  $\xi_i^{m'}$ ,  $\chi_j^{m'',m'}$ ,  $c_{j_0,j_1,j}^{m'',m'}$ ,  $d_{j,j_1}^{m'',m'}$ ,  $\alpha_i^{m'}$ ,  $\beta_{i_1,i_2,i_3}^{m'',m'}$ ,  $\beta_{i_1,i_2,i_3,i_4}^{m''',m'',m'}$ ,  $\beta_{i_1,i_2,i_3}^{m''',m'',m'}$  for  $m'' (< m')$ ,  $m''' (\leq m') \in \mathbf{N}$  and a finite number of  $C^\omega$  maps from  $M_0 \times \dots \times \mathbf{R}^{(k_{m'-1}+1)^2}$  to  $\mathbf{R}^{(k_{m'}+1)^2}$ , namely  $g_j^{m'} = (g_{j,0}^{m'}, \dots, g_{j,k_{m'}}^{m'})$  such that the following conditions are satisfied:

- the blowing-up  $\pi_{m'} : M_{m'} \rightarrow M_{m'-1}$  is the restriction to  $M_{m'}$  of the projection  $M_{m'-1} \times \mathbf{R}^{(k_{m'-1}+1)^2} \rightarrow M_{m'-1}$ ;
- $\{h_j^{m'} : j\}$  are generators of  $I(C_{m'})$  in  $C^\omega(M_{m'})$ ;
- $\{\xi_i^{m'}(y^{m'}) : i\}$  are generators of  $I(\mathbf{P}(k_{m'}))$  in  $\mathbf{R}[y_{i,j}^{m'}]_{0 \leq i,j \leq k_{m'}} (\subset C^\omega(\mathbf{R}^{(k_{m'}+1)^2}))$ ;
- $\{g_j^{m'} : j\}$  are generators of the sheaf of relations of  $h_0^{m'}, \dots, h_{k_{m'}}^{m'}$  on  $M_{m'}$ ;
- $\{\chi_j^{m'',m'} : j\}$  are generators of  $I(Y_{m'',m'})$  in  $C^\omega(M_{m'})$ , where  $Y_{m'-1,m'} = \pi_{m'}^{-1}(C_{m'-1})$  and

$$Y_{m'',m'} = (\pi_{m''+1} \circ \dots \circ \pi_{m'})^{-1}(C_{m''}) - (\pi_{m''+2} \circ \dots \circ \pi_{m'})^{-1}(C_{m''+1}) - \dots - \pi_{m'}^{-1}(C_{m'-1})$$

for  $m'' < m' - 1$ ;  $\{\alpha_i^{m'} : i\} = \{y_{i_1,i_2}^{m'-1} h_{i_3}^{m'-1} - y_{i_3,i_1}^{m'-1} h_{i_2}^{m'-1}, \sum_j y_{j,i_1}^{m'-1} g_{i_2,j}^{m'-1}, \xi_i^{m'-1} : i_1, i_2, i_3, i\}$ ;

- $\{\alpha_i^{m'''} : m''' \leq m', i\}$  are generators of  $I(M_{m'})$  in  $C^\omega(M_0 \times \dots \times \mathbf{R}^{(k_{m'-1}+1)^2})$ , where we naturally regard  $h_i^{m'-1}, g_{i,j}^{m'-1}, \xi_i^{m'-1}$  and  $\alpha_i^{m'''}$  as functions on  $M_0 \times \dots \times \mathbf{R}^{(k_{m'-1}+1)^2}$ ;
- (1)  $\sum_j g_{i,j}^{m'} h_j^{m'} = \sum_{m''' \leq m'} \beta_{1,i_1,i}^{m''',m'} \alpha_{i_1}^{m'''}$  on  $M_0 \times \dots \times \mathbf{R}^{(k_{m'-1}+1)^2}$ ;
- (2)  $vif = \sum_j a_{i,j}^{m'} h_j^{m'} + \sum_{m''' \leq m'} \beta_{2,i_2,i}^{m''',m'} \alpha_{i_2}^{m'''}$  on  $M_0 \times \dots \times \mathbf{R}^{(k_{m'-1}+1)^2}$ ;
- (4)  $\chi_j^{m'',m'} h_{j_1}^{m'-1} = \sum_{j_0} c_{j_0,j_1,j}^{m'',m'} \chi_{j_0}^{m'',m'-1} + \sum_{m''' \leq m'} \beta_{4,i,j,j_1}^{m''',m'',m'} \alpha_i^{m'''}$  on  $M_0 \times \dots \times \mathbf{R}^{(k_{m'-1}+1)^2}$  for  $m'' < m' - 1$ ;
- (5)  $\chi_j^{m'',m'} = \sum_{j_1} d_{j,j_1}^{m'',m'} h_{j_1}^{m'} + \sum_{m''' \leq m'} \beta_{5,i,j}^{m''',m'',m'} \alpha_i^{m'''}$  on  $M_0 \times \dots \times \mathbf{R}^{(k_{m'-1}+1)^2}$  for  $m'' < m' - 1$  if  $C_{m'} \subset Y_{m'',m'}$ ;
- (6)  $\chi_j^{m'-1,m'} = h_j^{m'-1}$  on  $M_0 \times \dots \times \mathbf{R}^{(k_{m'-1}+1)^2}$ .

(Condition (3) is included in (5) and (6).) Here  $d_{j,j_1}^{m'',m'}$  and  $\beta_{5,i,j}^{m''',m'',m'}$  exist only if  $C_{m'} \subset Y_{m'',m'}$  and we assume that if  $C_{m'} \not\subset Y_{m'',m'}$  then  $C_{m'}$  is transversal to  $Y_{m'',m'}$  in  $M_{m'}$ . Note that  $\cup_{m'' < m'} Y_{m'',m'}$  is a decomposition of  $\pi_{m'}^{-1}(C_{m'-1}) \cup \dots \cup (\pi_1 \circ \dots \circ \pi_{m'})^{-1}(C_0)$  into smooth analytic sets.

Assume, inductively, that there exist a blowing-up  $M_0 \times \dots \times \mathbf{R}^{(k_{m'-1}+1)^2} \supset \tilde{M}_{m'} \xrightarrow{\tilde{\pi}_{m'}} \tilde{M}_{m'-1}$  along center  $\tilde{C}_{m'-1}$  close to  $M_0 \times \dots \times \mathbf{R}^{(k_{m'-1}+1)^2} \supset M_{m'} \xrightarrow{\pi_{m'}} M_{m'-1}$  in the strong Whitney  $C^\infty$  topology and  $C^\omega$  approximations

$$\tilde{h}_j^{m'}, \tilde{a}_{i,j}^{m'}, \tilde{\chi}_j^{m'',m'}, \tilde{c}_{j_0,j_1,j}^{m'',m'}, \tilde{d}_{j,j_1}^{m'',m'}, \tilde{\alpha}_i^{m'}, \tilde{\beta}_{i_1,i_2,i_3}^{m'',m'}, \tilde{\beta}_{i_1,i_2,i_3,i_4}^{m''',m'',m'}, \tilde{\beta}_{i_1,i_2,i_3}^{m''',m'',m'}, \tilde{g}_j^{m'} = (\tilde{g}_{j,0}^{m'}, \dots, \tilde{g}_{j,k_{m'}}^{m'})$$

of

$$h_j^{m'}, a_{i,j}^{m'}, \chi_j^{m'',m'}, c_{j_0,j_1,j}^{m'',m'}, d_{j,j_1}^{m'',m'}, \alpha_i^{m'}, \beta_{i_1,i_2,i_3}^{m'',m'}, \beta_{i_1,i_2,i_3,i_4}^{m''',m'',m'}, \beta_{i_1,i_2,i_3}^{m''',m'',m'}, g_j^{m'} = (g_{j,0}^{m'}, \dots, g_{j,k_{m'}}^{m'})$$

on  $M_0 \times \dots \times \mathbf{R}^{(k_{m'-1}+1)^2}$  in the strong Whitney  $C^\infty$  topology such that  $\tilde{\pi}_{m'}$  is the restriction to  $\tilde{M}_{m'}$  of the projection  $M_0 \times \dots \times \mathbf{R}^{(k_{m'-1}+1)^2} \rightarrow M_0 \times \dots \times \mathbf{R}^{(k_{m'-2}+1)^2}$ ,

$$\{\tilde{\alpha}_i^{m'} : i\} = \{y_{i_1,i_2}^{m'-1} \tilde{h}_{i_3}^{m'-1} - y_{i_3,i_1}^{m'-1} \tilde{h}_{i_2}^{m'-1}, \sum_j y_{j,i_1}^{m'-1} \tilde{g}_{i_2,j}^{m'-1}, \xi_i^{m'-1} : i_1, i_2, i_3, i\}$$

and the corresponding conditions (1̃), (2̃), (4̃), (5̃) (if  $C_{m'} \subset Y_{m'',m'}$ ) and (6̃) to (1), (2), (4), (5) (if  $C_{m'} \subset Y_{m'',m'}$ ) and (6) are satisfied. Set  $\tilde{C}_{m'} = Z(\tilde{h}_j^{m'}) \cap \tilde{M}_{m'}$ ,  $\tilde{Y}_{m'-1,m'} = \tilde{\pi}_{m'}^{-1}(\tilde{C}_{m'-1})$  and

$$\tilde{Y}_{m'',m'} = (\tilde{\pi}_{m''-1} \circ \dots \circ \tilde{\pi}_{m'})^{-1}(\tilde{C}_{m''}) - (\tilde{\pi}_{m''} \circ \dots \circ \tilde{\pi}_{m'})^{-1}(\tilde{C}_{m''+1}) - \dots - \tilde{\pi}_{m'}^{-1}(\tilde{C}_{m'-1})$$

for  $m'' < m' - 1$ . Then, as above, we have:



- $I(\tilde{M}_{m'})$  in  $C^\omega(M_0 \times \dots \times \mathbf{R}^{(k_{m'-1}+1)^2})$  is generated by  $\{\tilde{\alpha}_i^{m'} : m'' \leq m', i\}$ ;
- $\tilde{C}_{m'}$  is smooth and of the same dimension as  $C_{m'}$ ;
- $I(\tilde{C}_{m'})$  in  $C^\omega(\tilde{M}_{m'})$  is generated by  $\{\tilde{h}_j^{m'} : j\}$ ;
- $\{\tilde{g}_j^{m'} : j\}$  are generators of the sheaf of relations of  $\tilde{h}_0^{m'}, \dots, \tilde{h}_{k_{m'}}^{m'}$  on  $\tilde{M}_{m'}$ ;
- $I(\tilde{Y}_{m'',m'})$  in  $C^\omega(\tilde{M}_{m'})$  for each  $m'' < m'$  is generated by  $\{\tilde{\chi}_j^{m'',m'} : j\}$  by (4);
- $\tilde{C}_{m'} \subset \tilde{Y}_{m'',m'}$  if and only if  $C_{m'} \subset Y_{m'',m'}$ ;
- if  $C_{m'} \not\subset Y_{m'',m'}$  then  $\tilde{C}_{m'}$  is transversal to  $\tilde{Y}_{m'',m'}$  in  $\tilde{M}_{m'}$ ;
- $\cup_{m'' < m'} \tilde{Y}_{m'',m'}$  is a decomposition of  $\tilde{\pi}_{m'}^{-1}(\tilde{C}_{m'-1}) \cup \dots \cup (\tilde{\pi}_1 \circ \dots \circ \tilde{\pi}_{m'})^{-1}(\tilde{C}_0)$  into smooth analytic sets;
- $\tilde{C}_{m'}$  is normal crossing with this set;
- there exists a  $C^\omega$  diffeomorphism from  $M_{m'}$  to  $\tilde{M}_{m'}$  close to id in the strong Whitney  $C^\infty$  topology and carrying  $C_{m'}, \dots, (\pi_1 \circ \dots \circ \pi_{m'})^{-1}(C_0)$  to  $\tilde{C}_{m'}, \dots, (\tilde{\pi}_1 \circ \dots \circ \tilde{\pi}_{m'})^{-1}(\tilde{C}_0)$ , respectively;
- $f \circ \tilde{\pi}_1 \circ \dots \circ \tilde{\pi}_{m'}$  is close to  $f \circ \pi_1 \circ \dots \circ \pi_{m'}$  in the strong Whitney  $C^\infty$  topology.

Finally, as above, we embed  $M_m$  and  $\tilde{M}_m$  into  $M_0 \times \mathbf{R}^{(k_0+1)^2} \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2}$  by  $h_0^{m-1}, \dots, h_{k_{m-1}}^{m-1}$  and  $\tilde{h}_0^{m-1}, \dots, \tilde{h}_{k_{m-1}}^{m-1}$ , respectively, define  $\alpha_i^m, \tilde{\alpha}_i^m, Y_{m',m}$  and  $\tilde{Y}_{m',m}$  for  $0 \leq m' < m$ , and let  $\{\chi_j^{m',m} : j\}$  and  $\{\tilde{\chi}_j^{m',m} : j\}$  be finitely many  $C^\omega$  functions on  $M_0 \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2}$  which are generators of  $I(Y_{m',m})$  in  $C^\omega(M_m)$  and of  $I(\tilde{Y}_{m',m})$  in  $C^\omega(\tilde{M}_m)$ , respectively, for each  $m' < m$  such that each  $\tilde{\chi}_j^{m',m}$  is close to  $\chi_j^{m',m}$  in the strong Whitney  $C^\infty$  topology. Then there exists a  $C^\omega$  diffeomorphism  $\psi_m : M_m \rightarrow \tilde{M}_m$  close to id in the strong Whitney  $C^\infty$  topology carrying  $\pi_m^{-1}(C_{m-1}), \dots, (\pi_1 \circ \dots \circ \pi_m)^{-1}(C_0)$  to  $\tilde{\pi}_m^{-1}(\tilde{C}_{m-1}), \dots, (\tilde{\pi}_1 \circ \dots \circ \tilde{\pi}_m)^{-1}(\tilde{C}_0)$ , respectively. Set  $F = f \circ \pi_1 \circ \dots \circ \pi_m$  and  $\tilde{F} = f \circ \tilde{\pi}_1 \circ \dots \circ \tilde{\pi}_m$ . Then  $F$  has only normal crossing singularities. We require  $\psi_m$  to carry, moreover,  $\text{Sing } F$  to  $\text{Sing } \tilde{F}$ . That is possible if  $\tilde{F}$  has only normal crossing singularities by the same reason as before.

We will describe a condition for  $\tilde{F}$  to have only normal crossing singularities. As the problem in the theorem is local around the compact subset  $X$  of  $M$ , we assume that  $M_m$  is covered by a finite number of good open subsets in the following sense. We have the disjoint union  $B$  of finitely many closed balls  $B_i$  in the Euclidean space of same dimension as  $M$ , a  $C^\omega$  immersion  $\rho = (\rho_{-1}, \dots, \rho_{m-1}) : B \rightarrow M_0 \times \mathbf{R}^{(k_0+1)^2} \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2}$ , finitely many  $C^\omega$  functions  $\delta_{i,j}$  on each  $B_i$  regular at  $\delta_{i,j}^{-1}(0)$  and  $\theta_{i,j} > 0 \in \mathbf{N}$  such that  $\text{Im } \rho \subset M_m$ ,  $\rho(\text{Int } B) \supset X$ , for each  $i$   $\rho|_{B_i}$  is an embedding,  $F \circ \rho|_{\text{Int } B_i}$  has only normal crossing singularities with unique singular value  $z_{0i}$ , and

$$F \circ \rho|_{B_i} = \prod_j \delta_{i,j}^{\theta_{i,j}} + z_{0i}.$$

Here the condition  $\text{Im } \rho \subset M_m$  and the last condition are equivalent to

$$(7) \quad f \circ \rho_{-1}|_{B_i} = \prod_j \delta_{i,j}^{\theta_{i,j}} + z_{0i}$$

and there exist  $C^\omega$  functions  $\nu_{i,j}^{m'}$  on  $M_0 \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B$  such that for each  $\alpha_i^{m'}$  with  $m' \leq m$

$$(8) \quad \alpha_i^{m'}(x, y^0, \dots, y^{m'-1}) = \nu_{i,-1}^{m'}(x, y^0, \dots, y^{m-1}, z)(x - \rho_{-1}(z)) +$$

$$\sum_{j=0}^{m-1} \nu_{i,j}^{m'}(x, y^0, \dots, y^{m-1}, z)(y^j - \rho_j(z)) \text{ on } M_0 \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B$$

because  $x - \rho_{-1}(z)$ ,  $y^j - \rho_j(z)$ ,  $j = 0, \dots, m-1$ , generate the ideal of  $C^\omega(M_0 \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B)$  defined by the graph of  $\rho - \{(\rho(z), z) : z \in B\}$ . Conversely, the existence of such  $\rho$ ,  $\delta_{i,j}$ ,  $\theta_{i,j}$  and  $\nu_{i,j}^{m'}$  implies the normal crossing property of  $F$ . Note that

$$\{z_{0i}\} = F(\text{Sing } F|_{(\pi_1 \circ \dots \circ \pi_m)^{-1}(U)}) = f(\text{Sing } f|_U)$$

for an open neighborhood  $U$  of  $X$  in  $M$ . (Assume that  $U = M$  for simplicity of notation.) Hence a condition for  $\tilde{F}$  to have only normal crossing singularities is that there exist  $C^\omega$  approximations  $\tilde{\rho} = (\tilde{\rho}_{-1}, \dots, \tilde{\rho}_{m-1}) : B \rightarrow M_0 \times \mathbf{R}^{(k_0+1)^2} \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2}$  of  $\rho$ ,  $\tilde{\delta}_{i,j}$  of  $\delta_{i,j}$  and  $\tilde{\nu}_{i,j}^{m'}$  of  $\nu_{i,j}^{m'}$  in the strong Whitney  $C^\infty$  topology such that

$$(7) \quad f \circ \tilde{\rho}_{-1}|_{B_i} = \prod_j \tilde{\delta}_{i,j}^{\theta_{i,j}} + z_{0i},$$

$$(8) \quad \tilde{\alpha}_i^{m'}(x, y^0, \dots, y^{m-1}) = \tilde{\nu}_{i,-1}^{m'}(x, y^0, \dots, y^{m-1}, z)(x - \tilde{\rho}_{-1}(z)) +$$

$$\sum_{j=0}^{m-1} \tilde{\nu}_{i,j}^{m'}(x, y^0, \dots, y^{m-1}, z)(y^j - \tilde{\rho}_j(z)) \quad \text{on } M_0 \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B.$$

However, we cannot find the approximations directly by proposition 3.1 below. Indeed, we need additional arguments as follows. Extend trivially  $\rho$  to  $\rho = (\rho_{-1}, \dots, \rho_{m-1}) : M_0 \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B \rightarrow M_0 \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2}$  and  $\delta_{i,j}$  to  $\delta_{i,j} : M_0 \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B_i \rightarrow \mathbf{R}$ . Then (7) holds on  $M_0 \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B_i$ . Approximate these extended  $\rho$  and  $\delta_{i,j}$  by a  $C^\omega$  map  $\tilde{\rho} = (\tilde{\rho}_{-1}, \dots, \tilde{\rho}_{m-1}) : M_0 \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B \rightarrow M_0 \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2}$  and  $C^\omega$  functions  $\tilde{\delta}_{i,j}$  on  $M_0 \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B_i$ , respectively, so that (7) and (8) hold on  $M_0 \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B$  and  $M_0 \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B_i$ , respectively. Regard  $M_0$  locally as a Euclidean space, and consider the map  $\tilde{P} : M_0 \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B \ni (x, y^0, \dots, y^{m-1}, z) \rightarrow (x - \tilde{\rho}_{-1}(x, \dots, z), \dots, y^{m-1} - \tilde{\rho}_{m-1}(x, \dots, z)) \in M_0 \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2}$ . As  $\tilde{P}$  is close to the map  $: M_0 \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2} \times B \ni (x, y^0, \dots, y^{m-1}, z) \rightarrow (x - \rho_{-1}(x, \dots, z), \dots, y^{m-1} - \rho_{m-1}(x, \dots, z)) \in M_0 \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2}$ , the Jacobian matrix  $\left| \frac{D(\tilde{P})}{D(x, \dots, y^{m-1})} \right|$  vanishes nowhere. Hence by the implicit function theorem we have an analytic map  $\hat{\rho} = (\hat{\rho}_{-1}, \dots, \hat{\rho}_{m-1}) : B \rightarrow M_0 \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2}$  such that  $\tilde{\rho}(\hat{\rho}(z), z) = \hat{\rho}(z)$  and  $\hat{\rho}$  is close to  $\rho$  in the strong Whitney topology. Then  $\hat{\rho}$  is a  $C^\omega$  immersion,

$$(7) \quad f \circ \hat{\rho}_{-1}(z) = f \circ \tilde{\rho}_{-1}(\hat{\rho}(z), z) = \prod_j \tilde{\delta}_{i,j}^{\theta_{i,j}}(\hat{\rho}(z), z) + z_{0i} \quad \text{for } z \in B_i,$$

$$(8) \quad \tilde{\alpha}_i^{m'} \circ \hat{\rho}(z) = \tilde{\nu}_{i,-1}^{m'}(\hat{\rho}(z), z)(\hat{\rho}_{-1}(z) - \tilde{\rho}_{-1}(\hat{\rho}(z), z)) +$$

$$\sum_{j=0}^{m-1} \tilde{\nu}_{i,j}^{m'}(\hat{\rho}(z), z)(\hat{\rho}_j(z) - \tilde{\rho}_j(\hat{\rho}(z), z)) = 0 \quad \text{for } z \in B.$$

By (8),  $\text{Im } \hat{\rho} \subset \tilde{M}_m$ , hence  $\rho(\text{Int } B) \supset X$ , and by (7),  $\tilde{F}$  has only normal crossing singularities because  $\tilde{\delta}_{i,j}(\hat{\rho}(z), z)$  is close to  $\delta_{i,j}(z)$  in the strong Whitney  $C^\infty$  topology. Note that if  $\tilde{\rho}$  is of class Nash, so is  $\hat{\rho}$ .

Under the conditions (7) and (8),  $F$  and  $\tilde{F}$  are  $C^\omega$  right equivalent through a  $C^\omega$  diffeomorphism close to id in the strong Whitney  $C^\infty$  topology for the following reason. Since  $F$  and  $\tilde{F}$  have only normal crossing singularities, and since  $f \circ \rho_{-1}$  and  $f \circ \hat{\rho}_{-1}$  are  $C^\omega$  right equivalent by (7) and (7), we can modify  $\psi_m$  to carry  $\text{Sing } F$  to  $\text{Sing } \tilde{F}$  (cf. step 1 of the proof of theorem 3.1 in [6]). Replacing  $\tilde{F}$  with  $\tilde{F} \circ \psi_m$ , we assume that  $\tilde{M}_m = M_m$ ,  $\tilde{\pi}_m^{-1}(\tilde{C}_{m-1}) = \pi_m^{-1}(C_{m-1}), \dots, (\tilde{\pi}_1 \circ \dots \circ \tilde{\pi}_m)^{-1}(\tilde{C}_0) = (\pi_1 \circ \dots \circ \pi_m)^{-1}(C_0)$  and

$\text{Sing } F = \text{Sing } \tilde{F}$ . Let  $\kappa$  be a Nash function on  $\mathbf{R}$  with zero set  $\{z_{0i}\}$  and regular there. Then  $\kappa \circ F$  and  $\kappa \circ \tilde{F}$  satisfy the assumption of lemma 4.7 in [6]:

- they have the same sign at each point of  $M$ , only normal crossing singularities at  $(\kappa \circ F)^{-1}(0) = F^{-1}(F(\text{Sing } F))$  and the same multiplicity at each point of  $(\kappa \circ F)^{-1}(0)$ ,
- the  $C^\omega$  function on  $M_m$ , defined to be  $\kappa \circ \tilde{F} / \kappa \circ F$  on  $M_m - (\kappa \circ F)^{-1}(0)$ , is close to 1 in the strong Whitney  $C^\infty$  topology. Indeed, the map  $:C^\omega(M_m) \ni \phi \rightarrow \phi \cdot (\kappa \circ F) \in \kappa \circ FC^\omega(M_m)$  is open in the strong Whitney  $C^\infty$  topology,  $\kappa \circ \tilde{F}$  is contained in  $\kappa \circ FC^\omega(M_m)$  and close to  $\kappa \circ F$  and hence there exist  $\phi \in C^\omega(M_m)$  close to 1 such that  $\phi \cdot (\kappa \circ F) = \kappa \circ \tilde{F}$ .

Therefore there exists a  $C^\omega$  diffeomorphism  $\psi'_m$  of  $M_m$  close to id in the strong Whitney  $C^\infty$  topology such that  $\psi'_m((\kappa \circ F)^{-1}(0)) = (\kappa \circ F)^{-1}(0)$  and  $F - \tilde{F} \circ \psi'_m$  is  $r$ -flat at  $(\kappa \circ F)^{-1}(0)$  for a large integer  $r$ . Then by proposition 4.8,(i) in [6],  $F$  and  $\tilde{F}$  are  $C^\omega$  right equivalent through a  $C^\omega$  diffeomorphism close to id in the strong Whitney  $C^\infty$  topology.

Consider the case of germ on  $X$ . Enlarging  $X$  if necessary we assume that  $X$  is semialgebraic. Set  $X_0 = X$ . Let  $h_j^{m'}$ ,  $g_j^{m'}$ ,  $a_{i,j}^{m'}$ ,  $\dots$ ,  $\nu_{i,j}^{m'}$  be the same as above. Let  $\tilde{h}_j^0$ ,  $\tilde{g}_j^0$ ,  $\tilde{a}_{i,j}^0$  be defined not on  $M_0$  but on an open neighborhood  $U_0$  of  $X_0$  in  $M_0$  close to  $h_j^0$ ,  $g_j^0$ ,  $a_{i,j}^0$ , respectively, at  $X_0$  in the  $C^\infty$  topology so that  $(\tilde{1})$  and  $(\tilde{2})$  hold on  $U_0$ . Shrink  $U_0$  if necessary. Then by remark 2.3.(2) of lemma 2.2 we have the blowing-up  $U_0 \times \mathbf{R}^{(k_0+1)^2} \supset U_1 \xrightarrow{\tau_1} U_0$  along center  $D_0 = Z(\tilde{h}_j^0)$  defined by  $\tilde{h}_0^0, \dots, \tilde{h}_{k_0}^0$  and analytic embeddings  $\psi_0$  of  $U_0$  into  $M_0$  and  $\tilde{\psi}_0$  of  $U_1$  into  $M_1$  close to id at  $X_0$  and at  $\tau_1(X)$ , respectively, such that  $\psi_0(D_0) \subset C_0$  and  $\psi_0 \circ \tau_1 = \pi_1 \circ \tilde{\psi}_0$ .

Next let  $\tilde{h}_j^1$ ,  $\tilde{g}_j^1$ ,  $\tilde{a}_{i,j}^1$ ,  $\tilde{b}_{j_0,j_1}^1$ ,  $\tilde{\beta}_{i_1,i_2,i_3}^1$  be defined on an open neighborhood of  $X_0 \times X_1$  in  $M_0 \times \mathbf{R}^{(k_0+1)^2}$  close to  $h_j^1, \dots, \beta_{i_1,i_2,i_3}^1$ , respectively, at  $X_0 \times X_1$  in the  $C^\infty$  topology such that  $(\tilde{1})$ ,  $(\tilde{2})$  and  $(\tilde{3})$  hold on the neighborhood, where  $\tilde{\alpha}_i^1$  are defined as in the global case and  $X_1$  denotes a large ball in  $\mathbf{R}^{(k_0+1)^2}$  with center 0 such that  $\pi_1^{-1}(X_0)$  and  $\tau_1^{-1}(X_0)$  are contained in  $X_0 \times \text{Int } X_1$ . Shrink  $U_0$  and the neighborhood of  $X_0 \times X_1$  so that  $U_1$  and  $M_1 \cap U_0 \times \mathbf{R}^{(k_0+1)^2}$  are closed subsets of the neighborhood; it is possible because  $\pi_1$  and  $\tau_1$  are proper. Then there exist the blowing-up  $U_0 \times \mathbf{R}^{(k_0+1)^2} \times \mathbf{R}^{(k_1+1)^2} \supset U_2 \xrightarrow{\tau_2} U_1$  along center  $D_1 = Z(\tilde{h}_j^1) \cap U_1$  defined by  $\tilde{h}_0^1, \dots, \tilde{h}_{k_1}^1$  and analytic embeddings  $\psi_1$  of  $U_1$  into  $M_1$  and  $\tilde{\psi}_1$  of  $U_2$  into  $M_2$  close to id at  $\tau_1^{-1}(X_0)$  and at  $(\tau_1 \circ \tau_2)^{-1}(X_0)$ , respectively, such that  $\psi_1(D_1) \subset C_1$  and  $\psi_1 \circ \tau_2 = \pi_2 \circ \tilde{\psi}_1$ .

Let  $1 < m' < m$ ,  $m'' < m'$  and  $m''' \leq m'$ . By induction, let  $\tilde{h}_j^{m'}$ ,  $\tilde{g}_j^{m'}$ ,  $\tilde{a}_{i,j}^{m'}$ ,  $\tilde{\chi}_{j_0,j_1,j}^{m',m'}$ ,  $\tilde{c}_{j_0,j_1,j}^{m'',m'}$ ,  $\tilde{d}_{j,j_1}^{m'',m'}$ ,  $\tilde{\beta}_{i_1,i_2,i_3}^{m'',m'}$ ,  $\tilde{\beta}_{i_1,i_2,i_3,i_4}^{m''',m'',m'}$ ,  $\tilde{\beta}_{i_1,i_2,i_3}^{m''',m'',m'}$  be defined on an open neighborhood of  $X_0 \times \dots \times X_{m'}$  in  $M_0 \times \mathbf{R}^{(k_0+1)^2} \times \dots \times \mathbf{R}^{(k_{m'}+1)^2}$  close to  $h_j^{m'}$ ,  $g_j^{m'}$ ,  $\dots$ , respectively, at  $X_0 \times \dots \times X_{m'}$  in the  $C^\infty$  topology such that  $(\tilde{1})$ ,  $(\tilde{2})$ ,  $(\tilde{4})$ ,  $(\tilde{5})$ ,  $(\tilde{6})$  hold on the neighborhood, where  $\tilde{\alpha}_i^{m'}$  are given as in the global case and  $X_i$  denotes a large ball in  $\mathbf{R}^{(k_{i-1}+1)^2}$  with center 0 for  $i = 2, \dots, m'$ . For  $m' (\leq m) \in \mathbf{N}$ , let  $\tilde{\alpha}_i^{m'}$  and  $\tilde{\chi}_j^{m',m'}$  be defined on an open neighborhood of  $X_0 \times \dots \times X_m$  close to  $\alpha_i^{m'}$  and  $\chi_j^{m',m'}$ , respectively, at  $X_0 \times \dots \times X_m$  as before, and  $\tilde{\rho}_i$ ,  $\tilde{\delta}_{i,j}$ ,  $\tilde{\nu}_{i,j}^{m'}$  on an open neighborhood of  $X_0 \times \dots \times X_m \times B$  close to  $\rho_i$ ,  $\delta_{i,j}$ ,  $\nu_{i,j}^{m'}$ , respectively, at  $X_0 \times \dots \times X_m \times B$  so that  $(\tilde{7})$  and  $(\tilde{8})$  hold on the neighborhood.

Then we obtain a sequence of blowings-up  $U_m \xrightarrow{\tau_m} U_{m-1} \rightarrow \dots \xrightarrow{\tau_1} U_0$  along smooth analytic centers  $D_{m-1} = Z(\tilde{h}_j^{m-1}) \cap U_{m-1}$  in  $U_{m-1}, \dots, D_0 = Z(\tilde{h}_j^0)$  in  $U_0$ , respectively, and an analytic embedding  $\psi : U_m \rightarrow M_m$  such that  $\psi(\tau_m^{-1}(D_{m-1})) \subset \pi_m^{-1}(C_{m-1}), \dots, \psi((\tau_1 \circ \dots \circ \tau_m)^{-1}(D_0)) \subset (\pi_1 \circ \dots \circ \pi_m)^{-1}(C_0)$ ,  $f \circ \pi_1 \circ \dots \circ \pi_m \circ \psi = f \circ \tau_1 \circ \dots \circ \tau_m$ ,  $U_1, \dots, U_m$  are realized in  $U_0 \times \mathbf{P}(k_0) \subset U_0 \times \mathbf{R}^{(k_0+1)^2}, \dots, U_0 \times \mathbf{P}(k_0) \times \dots \times \mathbf{P}(k_{m-1}) \subset U_0 \times$

$\mathbf{R}^{(k_0+1)^2} \times \dots \times \mathbf{R}^{(k_{m-1}+1)^2}$ , respectively, and each pair  $D_i \subset U_i$  and  $\psi$  are close to  $C_i \subset M_i$  at  $(\tau_1 \circ \dots \circ \tau_{i-1})^{-1}(X_0)$  and to  $\text{id}$  at  $(\tau_1 \circ \dots \circ \tau_m)^{-1}(X_0)$ , respectively, in the  $C^\infty$  topology.

Thus it remains only to find the approximations  $\tilde{h}_j^0, \tilde{g}_j^0, \dots$  of class Nash. This is a consequence of proposition 3.1 below.

□

### 3. NESTED NASH APPROXIMATION

**3.1. Nash approximation of an analytic diffeomorphism.** In order to prove theorem 3.2 below and theorem 1.11, we need to make a Nash approximation of analytic solutions of a system of Nash equations. The following proposition is a nested version of the Nash Approximation Theorem established in [3].

**Proposition 3.1.** *Let  $M_1, \dots, M_m$  be Nash manifolds,  $X_1 \subset M_1, \dots, X_m \subset M_m$  compact semialgebraic subsets, and let  $l_1, \dots, l_m, n_1, \dots, n_m \in \mathbf{N}$ . Let  $F_i \in \mathcal{N}(X_1 \times \dots \times X_i \times \mathbf{R}^{l_1} \times \dots \times \mathbf{R}^{l_i})^{n_i}$  and  $f_i \in \mathcal{O}(X_1 \times \dots \times X_i)^{l_i}$  for  $i = 1, \dots, m$  such that*

$$F_i(x_1, \dots, x_i, f_1(x_1), \dots, f_i(x_1, \dots, x_i)) = 0$$

*as elements of  $\mathcal{O}(X_1 \times \dots \times X_i)^{n_i}$ . Then there exist  $\tilde{f}_i \in \mathcal{N}(X_1 \times \dots \times X_i)^{l_i}$  close to  $f_i$  in the  $C^\infty$  topology for  $i = 1, \dots, m$  such that  $F_i(x_1, \dots, x_i, \tilde{f}_1(x_1), \dots, \tilde{f}_i(x_1, \dots, x_i)) = 0$  in  $\mathcal{N}(X_1 \times \dots \times X_i)^{n_i}$ .*

*Proof.* The proof is inspired by Nested Smoothing Theorem 11.4, [21] by Teissier and its proof. The proof for  $m = 1$  coincides with Theorem 1.1, [3] if  $M_1$  is compact and if  $X_1 = M_1$ , and we can prove the proposition for  $m = 1$  in the same way even if  $M_1$  is non-compact.

Regard each  $F_i$  as  $n_i$  elements of  $\mathcal{N}(X_1 \times \dots \times X_i \times \mathbf{R}^{l_1} \times \dots \times \mathbf{R}^{l_i})$ . We can assume that  $M_i$  and  $X_i$  are all connected and that  $F_i$  are polynomial functions in the variables  $(y_1, \dots, y_i) \in \mathbf{R}^{l_1} \times \dots \times \mathbf{R}^{l_i}$  with coefficients in  $\mathcal{N}(X_1 \times \dots \times X_i)$  for the same reason as in the proof of Theorem 1.1, [3]. Let  $\mathcal{N}(X_1 \times \dots \times X_i)[y_1, \dots, y_i]$  denote the ring of polynomials in the variables  $(y_1, \dots, y_i) \in \mathbf{R}^{l_1} \times \dots \times \mathbf{R}^{l_i}$  with coefficients in  $\mathcal{N}(X_1 \times \dots \times X_i)$  and  $(F_1, \dots, F_i)$  the ideal of  $\mathcal{N}(X_1 \times \dots \times X_i)[y_1, \dots, y_i]$  generated by  $F_1, \dots, F_i$ .

Consider a commutative diagram of homomorphisms between rings:

$$\begin{array}{ccccc} \mathcal{N}(X_1) & \xrightarrow{\phi_1} & \frac{\mathcal{N}(X_1)[y_1]}{(F_1)} & \xrightarrow{\psi_1} & \mathcal{O}(X_1) \\ \downarrow p_1 & & \downarrow q_1 & & \downarrow r_1 \\ \mathcal{N}(X_1 \times X_2) & \xrightarrow{\phi_2} & \frac{\mathcal{N}(X_1 \times X_2)[y_1, y_2]}{(F_1, F_2)} & \xrightarrow{\psi_2} & \mathcal{O}(X_1 \times X_2) \\ \downarrow p_2 & & \downarrow q_2 & & \downarrow r_2 \\ \vdots & & \vdots & & \vdots \\ \downarrow p_{m-1} & & \downarrow q_{m-1} & & \downarrow r_{m-1} \\ \mathcal{N}(X_1 \times \dots \times X_m) & \xrightarrow{\phi_m} & \frac{\mathcal{N}(X_1 \times \dots \times X_m)[y_1, \dots, y_m]}{(F_1, \dots, F_m)} & \xrightarrow{\psi_m} & \mathcal{O}(X_1 \times \dots \times X_m), \end{array}$$

where for each  $i$ ,  $\phi_i, p_i, q_i$  and  $r_i$  are naturally defined,  $\psi_i = \text{id}$  on  $\mathcal{N}(X_1 \times \dots \times X_i)$  and  $\psi_i(y_j)$  is defined to be  $f_j$  as an element of  $\mathcal{O}(X_1 \times \dots \times X_i)$  for each  $j \leq i$ . Then it suffices to find homomorphisms  $\tilde{\psi}_1 : \mathcal{N}(X_1)[y_1]/(F_1) \rightarrow \mathcal{N}(X_1), \dots, \tilde{\psi}_m : \mathcal{N}(X_1 \times \dots \times X_m)[y_1, \dots, y_m]/(F_1, \dots, F_m) \rightarrow \mathcal{N}(X_1 \times \dots \times X_m)$  such that  $\tilde{\psi}_1 \circ \phi_1 = \text{id}, \dots, \tilde{\psi}_m \circ \phi_m = \text{id}$ ,  $\tilde{\psi}_1(y_1), \dots, \tilde{\psi}_m(y_m)$  are close to  $f_1, \dots, f_m$ , respectively, and  $r_i \circ \tilde{\psi}_i = \tilde{\psi}_{i+1} \circ q_i$  for  $0 < i < m$ .

For that we only need to decide the values  $\tilde{\psi}_1(y_1), \dots, \tilde{\psi}_m(y_m)$  because  $\tilde{\psi}_i(y_k) = \tilde{\psi}_k(y_k)$  as elements of  $\mathcal{N}(X_1 \times \dots \times X_i)$  for  $k < i$  by the equality  $r_j \circ \tilde{\psi}_j = \tilde{\psi}_{j+1} \circ q_j$ . By [4], [7] and [18] we know that  $\mathcal{O}(X_1 \times \dots \times X_i)$  and  $N(M_1 \times \dots \times M_i)$  are Noetherian, and the proofs in [4] and [18] work for the Noetherian property of  $\mathcal{N}(X_1 \times \dots \times X_i)$ . Hence all the rings in the diagram are Noetherian. Therefore, we assume that  $\psi_i$  are injective, adding some Nash functions to  $F_i$ 's if necessary.

We will find  $k_i \in \mathbf{N}$ , finite subsets  $G_i$  of  $\mathcal{N}(X_1 \times \dots \times X_i)[z_1, \dots, z_i]$ ,  $z_j \in \mathbf{R}^{k_j}$ , and a commutative diagram of homomorphisms between rings:

$$\begin{array}{ccccc}
\frac{\mathcal{N}(X_1)[y_1]}{(F_1)} & \xrightarrow{\rho_1} & \frac{\mathcal{N}(X_1)[z_1]}{(G_1)} & \xrightarrow{\xi_1} & \mathcal{O}(X_1) \\
\downarrow q_1 & & \downarrow s_1 & & \downarrow r_1 \\
\frac{\mathcal{N}(X_1 \times X_2)[y_1, y_2]}{(F_1, F_2)} & \xrightarrow{\rho_2} & \frac{\mathcal{N}(X_1 \times X_2)[z_1, z_2]}{(G_1, G_2)} & \xrightarrow{\xi_2} & \mathcal{O}(X_1 \times X_2) \\
\downarrow q_2 & & \downarrow s_2 & & \downarrow r_2 \\
\vdots & & \vdots & & \vdots \\
\downarrow q_{m-1} & & \downarrow s_{m-1} & & \downarrow r_{m-1} \\
\frac{\mathcal{N}(X_1 \times \dots \times X_m)[y_1, \dots, y_m]}{(F_1, \dots, F_m)} & \xrightarrow{\rho_m} & \frac{\mathcal{N}(X_1 \times \dots \times X_m)[z_1, \dots, z_m]}{(G_1, \dots, G_m)} & \xrightarrow{\xi_m} & \mathcal{O}(X_1 \times \dots \times X_m)
\end{array}$$

such that for each  $i$ ,  $s_i$  is naturally defined,  $\xi_i \circ \rho_i = \psi_i$ ,  $\rho_i = \xi_i = \text{id}$  on  $\mathcal{N}(X_1 \times \dots \times X_i)$ , the zero set  $Z_i$  of  $(G_1, \dots, G_i)$  is the germ on  $X_1 \times \dots \times X_i \times \mathbf{R}^{k_1} \times \dots \times \mathbf{R}^{k_i}$  of a Nash submanifold of  $M_1 \times \dots \times M_i \times \mathbf{R}^{k_1} \times \dots \times \mathbf{R}^{k_i}$  and  $(G_1, \dots, G_i)$  is the ideal of  $\mathcal{N}(X_1 \times \dots \times X_i)[z_1, \dots, z_i]$  of function germs vanishing on  $Z_i$ . Note that the restriction  $\pi_i$  to  $Z_i$  of the projection  $M_1 \times \dots \times M_i \times \mathbf{R}^{k_1} \times \dots \times \mathbf{R}^{k_i} \rightarrow M_1 \times \dots \times M_i$  is submersive because  $\mathcal{N}(X_1 \times \dots \times X_i) \subset \mathcal{N}(X_1 \times \dots \times X_i)[z_1, \dots, z_i]/(G_1, \dots, G_i)$  and  $\xi_i|_{\mathcal{N}(X_1 \times \dots \times X_i)} = \text{id}$ , that  $\xi_i(z_1, \dots, z_i)$  is an analytic cross-section of  $\pi_i$  and that when we regard  $M_j$  locally as Euclidean spaces the rank of the Jacobian matrix  $\frac{D(G_1, \dots, G_i)}{D(x_1, \dots, x_i, z_1, \dots, z_i)}$  equals the codimension of  $Z_i$  in  $M_1 \times \dots \times M_i \times \mathbf{R}^{k_1} \times \dots \times \mathbf{R}^{k_i}$  at each point of  $Z_i$ .

As in the proof of Theorem 1.1, [3] there exist  $k_i \in \mathbf{N}$ , finite subsets  $G_i$  of  $\mathcal{N}(X_1 \times \dots \times X_i)[z_i]$ ,  $z_i \in \mathbf{R}^{k_i}$  and homomorphisms of  $\mathcal{N}(X_1 \times \dots \times X_i)$ -algebras

$$\frac{\mathcal{N}(X_1 \times \dots \times X_i)[y_1, \dots, y_i]}{(F_1, \dots, F_i)} \xrightarrow{\rho'_i} \frac{\mathcal{N}(X_1 \times \dots \times X_i)[z_i]}{(G_i)} \xrightarrow{\xi'_i} \mathcal{O}(X_1 \times \dots \times X_i)$$

such that for each  $i$ ,  $\xi'_i \circ \rho'_i = \psi_i$ , the zero set  $Z'_i$  of  $(G_i)$  is the germ on  $X_1 \times \dots \times X_i \times \mathbf{R}^{k_i}$  of a Nash submanifold of  $M_1 \times \dots \times M_i \times \mathbf{R}^{k_i}$  and  $(G_i)$  is the ideal of  $\mathcal{N}(X_1 \times \dots \times X_i)[z_i]$  of function germs vanishing on  $Z'_i$ . Then as above the restriction  $\pi'_i$  to  $Z'_i$  of the projection  $M_1 \times \dots \times M_i \times \mathbf{R}^{k_i} \rightarrow M_1 \times \dots \times M_i$  is submersive and  $\xi'_i(z_i)$  is an analytic cross-section of  $\pi'_i$ . Define  $\rho_i$  to be the composition of  $\rho'_i$  with the canonical homomorphism  $\mathcal{N}(X_1 \times \dots \times X_i)[z_i]/(G_i) \rightarrow \mathcal{N}(X_1 \times \dots \times X_i)[z_1, \dots, z_i]/(G_1, \dots, G_i)$  and  $\xi_i$  by  $\xi_i(z_i) = \xi'_i(z_i)$  and  $\xi_i(z_j) = r_{i-1} \circ \dots \circ r_j \circ \xi'_j(z_j)$ . Then the conditions on  $G_i$ ,  $\rho_i$  and  $\xi_i$  are satisfied. Indeed, first the zero set  $Z_i$  of  $(G_1, \dots, G_i)$  in  $M_1 \times \dots \times M_i \times \mathbf{R}^{k_1} \times \dots \times \mathbf{R}^{k_i}$  is the fiber product of the submersions  $(\pi'_1, \text{id}) : Z'_1 \times M_2 \times \dots \times M_i \rightarrow M_1 \times \dots \times M_i$ ,  $(\pi'_2, \text{id}) : Z'_2 \times M_3 \times \dots \times M_i \rightarrow M_1 \times \dots \times M_i, \dots, \pi'_i : Z'_i \rightarrow M_1 \times \dots \times M_i$  and hence the germ on  $X_1 \times \dots \times X_i \times \mathbf{R}^{k_1} \times \dots \times \mathbf{R}^{k_i}$  of some Nash submanifold of  $M_1 \times \dots \times M_i \times \mathbf{R}^{k_1} \times \dots \times \mathbf{R}^{k_i}$ . Next, add some finite subset of  $\mathcal{N}(X_1 \times \dots \times X_i)[z_1, \dots, z_i]$  to  $G_i$  whose elements vanish on  $Z_i$ , if necessary. Then  $(G_1, \dots, G_i)$  is the ideal of  $\mathcal{N}(X_1 \times \dots \times X_i)[z_1, \dots, z_i]$  of function germs vanishing on  $Z_i$ . Thus we obtain the required diagram.

For the construction of  $\tilde{\psi}_i$ 's it suffices to find homomorphisms of  $\mathcal{N}(X_1 \times \cdots \times X_i)$ -algebras  $\tilde{\xi}_i : \mathcal{N}(X_1 \times \cdots \times X_i)[z_1, \dots, z_i]/(G_1, \dots, G_i) \rightarrow \mathcal{N}(X_1 \times \cdots \times X_i)$  so that  $\tilde{\xi}_i(z_i) \in \mathcal{N}(X_1 \times \cdots \times X_i)^{k_i}$  are close to  $\xi_i(z_i) \in \mathcal{O}(X_1 \times \cdots \times X_i)^{k_i}$  in the  $C^\infty$  topology because if we define  $\tilde{\xi}_i$  by  $\tilde{\xi}_i(z_j) = r_{i-1} \circ \cdots \circ r_j \circ \xi_j(z_j)$  for  $j < i$  then  $\tilde{\psi}_i = \tilde{\xi}_i \circ \rho_i$  fulfill the requirements. By induction on  $m$  we assume that  $\xi_1, \dots, \xi_{m-1}$  are given. Then as before we only need to decide  $\tilde{\xi}_m(z_m) \in \mathcal{N}(X_1 \times \cdots \times X_m)^{k_m}$  close to  $\xi_m(z_m) \in \mathcal{O}(X_1 \times \cdots \times X_m)^{k_m}$  in the  $C^\infty$  topology so that  $G_m(x_1, \dots, x_m, \tilde{\xi}_m(z_m)) = \{0\}$  as a subset of  $\mathcal{N}(X_1 \times \cdots \times X_m)$ , i.e.  $\tilde{\xi}_m(z_m)$  is a Nash cross-section of  $\pi'_m$ . (Here the elements of  $G_m$  may be of the variables  $x_1, \dots, x_m, z_1, \dots, z_m$ . However, we can remove some elements from  $G_m$  so that they are all in the variables  $x_1, \dots, x_m, z_m$  by the above arguments.) Let  $U \subset U'$  be small open semialgebraic neighborhoods of  $X_1 \times \cdots \times X_m$  in  $M_1 \times \cdots \times M_m$  such that  $\bar{U}$  is compact and contained in  $U'$ ,  $Z'_m$  is the germ on  $X_1 \times \cdots \times X_m \times \mathbf{R}^{k_m}$  of a closed Nash submanifold  $Z'$  of  $U' \times \mathbf{R}^{k_m}$ ,  $\pi'_m$  is the germ on  $X_1 \times \cdots \times X_m \times \mathbf{R}^{k_m}$  of a surjective submersion  $\pi' : Z' \rightarrow U'$  and  $\xi_m(z_m)$  is the germ on  $X_1 \times \cdots \times X_m$  of an analytic cross-section  $\xi : \bar{U} \rightarrow Z'$  of  $\pi'$ . Let  $\eta$  be a Nash approximation of  $\xi|_{\bar{U}} : \bar{U} \rightarrow Z'$  in the  $C^\infty$  topology (Nash Approximation Theorem), which is an embedding but not necessarily a cross-section of  $\pi'|_{\pi'^{-1}(\bar{U})}$ . Let  $\tilde{\xi}_m(z_m)$  be the germ of  $\eta \circ (\pi'_m \circ \eta)^{-1}$  on  $X_1 \times \cdots \times X_m$ . Then  $\tilde{\xi}_m(z_m)$  is a Nash cross-section of  $\pi'_m$  close to  $\xi_m(z_m)$  in the  $C^\infty$  topology. Thus we complete the proof.  $\square$

As a corollary of proposition 3.1 we obtain the following Nash approximation theorem, which generalizes that proved in [3] in the case where  $X = M$  and  $M$  is compact.

**Theorem 3.2.** *Let  $M$  be a Nash manifold,  $X \subset M$  be a compact semialgebraic subset, and  $f, g$  be Nash function germs on  $X$  in  $M$ . If  $f$  and  $g$  are analytically right equivalent, then  $f$  and  $g$  are Nash right equivalent. The diffeomorphism of Nash right equivalence can be chosen to be close to the given one of analytic right equivalence in the  $C^\infty$  topology.*

Here we naturally define analytic or Nash right equivalence of two analytic or Nash function germs, respectively, on  $X$  in  $M$ . We note only that the diffeomorphism germ of equivalence is  $X$ -preserving.

For the proof we introduce some notions. Let  $X$  be a semialgebraic subset of a Nash manifold  $M$ . We consider the germs of sets on  $X$  in  $M$ . For a germ  $A$  on  $X$  of a subset of  $M$ , let  $\bar{A}^X$  or  $A^{-X}$  denote the *Nash closure* of  $A$  in  $M$ , i.e. the smallest Nash set germ in  $M$  containing  $A$ . In the case where  $A$  is a subset of  $M$  also,  $\bar{A}^X$  coincides with the Nash closure of the germ of  $A$  on  $X$  in  $M$ . We define by induction a sequence of Nash set germs  $M_i$  in  $M$  as follows. Let  $M_1$  be the germ  $\bar{X}^X$  and assume that  $M_1, \dots, M_{k-1}$  are given for  $k (> 1) \in \mathbf{N}$ . Then, set

$$M_k = \overline{[(M_{k-1} - X) \cap (\bar{M}_{k-1} \cap X)]}^X.$$

We call  $\{M_i\}$  the *canonical Nash germ decomposition* of  $X$ . Then  $\{M_i\}$  is a decreasing sequence of Nash set germs, for each  $i$  the set  $X \cap M_i - M_{i+1}$  is a union of some connected components of  $M_i - M_{i+1}$  and  $\{M_i\}$  is canonical in the following sense. Let  $\{M'_i\}$  be another decreasing sequence of Nash set germs such that for each  $i$  the set  $X \cap M'_i - M'_{i+1}$  is a union of some connected components of  $M'_i - M'_{i+1}$ , which is called a *Nash germ decomposition* of  $X$ . Assume that  $\{M'_i\}$  is distinct from  $\{M_i\}$ . Then  $M'_1 = M_1, \dots, M'_{k-1} = M_{k-1}$  and  $M'_k \supsetneq M_k$  for some  $k$ .

A subset  $Y$  of an analytic manifold  $N$  is called *global semianalytic* if  $Y$  is described by finitely many equalities and inequalities of global analytic functions on  $N$ . Let  $Y$  be a relatively compact and global semianalytic subset of  $N$ . Then we can define the

global analytic closure  $\overline{Z}^Y$  of the germ on  $Y$  of a subset  $Z$  of  $N$  (or of the germ  $Z$  on  $Y$  of a subset of  $N$ ) and a (the canonical) global analytic germ decomposition of  $Y$  in the same way. Indeed, for a global semianalytic set  $Z$  in  $N$ ,  $\dim Z = \dim \overline{Z}^Y$  for the reason explained below, and if  $Z$  is, moreover, relatively compact then  $\overline{Z}$  is global semianalytic by [19] and finally, a global analytic set is global semianalytic (and moreover is the zero set of one global analytic function). To prove that  $\dim Z = \dim \overline{Z}^Y$  we can assume that  $Z$  is a global semianalytic set of the form  $\{x \in N : f(x) = 0, f_1 > 0, \dots, f_k(x) > 0\}$  for some analytic functions  $f, f_1, \dots, f_k$  on  $N$  dividing  $Z$  if necessary, and it suffices to prove that the global analytic closure  $\overline{Z}^N$  of  $Z$  is of the same dimension as  $Z$ . Let  $x_0 \in Z$  where the germ of  $\overline{Z}^N$  is of dimension  $\dim \overline{Z}^N$ . There exists such a point since  $Z \cap \text{Reg } \overline{Z}^N \neq \emptyset$ . Then  $f_1 > 0, \dots, f_k > 0$  on a neighborhood of  $x_0$  in  $\overline{Z}^N$ . Hence  $Z$  contains the neighborhood and is of dimension  $\dim \overline{Z}^N$ . (We do not know whether the canonical global analytic germ decomposition of  $Y$  exists if  $Y$  is a non-relatively compact global semianalytic set.)

**Remark 3.3.**

- (1) Let  $N \supset Y$  and  $N' \supset Y'$  be analytic manifolds and respective relatively compact and global semianalytic subsets and  $\phi : N \rightarrow N'$  an analytic diffeomorphism such that  $\phi(Y) = Y'$ . Then  $\phi$  carries the canonical global analytic germ decomposition of  $Y$  to the canonical global analytic germ decomposition of  $Y'$ .
- (2) Let  $M \supset X$  be a Nash manifold and a semialgebraic subset. Then the canonical global analytic germ decomposition of  $X$  is well-defined and coincides with the canonical Nash germ decomposition of  $X$  because the global analytic closure of a semialgebraic set equals its Nash closure.

*Proof of theorem 3.2.* Let  $M \subset \mathbf{R}^n$ , set  $M_0 = \overline{M}^X$ , and let  $\{M_i : i = 1, 2, \dots\}$  be the canonical Nash germ decomposition of  $X$ . Let  $\mathcal{O}(X)$  and  $\mathcal{N}(X)$  denote respectively the germs of analytic and Nash functions on  $X$  in  $\mathbf{R}^n$  but not in  $M$ . Let  $\{\phi_{i,j} : j\}$  for each  $i = 0, 1, \dots$  be finitely many generators of the ideal of  $\mathcal{N}(X)$  defined by  $M_i$ . Extend  $f$  and  $g$  to elements  $\hat{f}$  and  $\hat{g}$  of  $\mathcal{N}(X)$ , respectively. Then we have  $\pi = (\pi_1, \dots, \pi_n) \in \mathcal{O}(X)^n$  such that  $\pi|_M$  is the germ on  $X$  of a  $C^\omega$  diffeomorphism between neighborhoods of  $X$  in  $M$  and  $f \circ \pi = g$  on  $M$ . Hence there exist  $\alpha_j \in \mathcal{O}(X)$  such that

$$(1) \quad \hat{f} \circ \pi = \hat{g} + \sum_j \alpha_j \phi_{0,j}.$$

By remarks 3.3.(1) and 3.3.(2),  $\pi$  is  $M_i$ -preserving. Hence there exist  $\beta_{i,j,j'} \in \mathcal{O}(X)$  such that for each  $\phi_{i,j}$

$$(2) \quad \phi_{i,j} \circ \pi = \sum_{j'} \beta_{i,j,j'} \phi_{i,j'}.$$

Apply proposition 3.1 to (1) and (2). Then there exist  $\tilde{\pi} \in \mathcal{N}(X)^n$ ,  $\tilde{\alpha}_j \in \mathcal{N}(X)$  and  $\tilde{\beta}_{i,j,j'} \in \mathcal{N}(X)$  close to  $\pi, \alpha_j$  and  $\beta_{i,j,j'}$ , respectively, in the  $C^\infty$  topology such that

$$(\tilde{1}) \quad \hat{f} \circ \tilde{\pi} = \hat{g} + \sum_j \tilde{\alpha}_j \phi_{0,j},$$

$$(\tilde{2}) \quad \phi_{i,j} \circ \tilde{\pi} = \sum_{j'} \tilde{\beta}_{i,j,j'} \phi_{i,j'}.$$

Since  $\tilde{\pi}$  is an approximation of  $\pi$ , (2) implies that  $\tilde{\pi}|_M$  is the germ on  $X$  of a Nash diffeomorphism between open semialgebraic neighborhoods of  $X$  in  $M$ . Hence by (1),  $f \circ (\tilde{\pi}|_M) = g$ , and the theorem is proved.  $\square$

Consider the plural case of  $\{X\}$ . Let  $X$  and  $X_j$ ,  $j = 1, \dots, k$ , be semialgebraic subsets of a Nash manifold  $M$ . We define the *canonical Nash germ decomposition*  $\{M_i\}$  of  $\{X; X_j\}$  as follows. Set  $X_0 = \cup_{j=1}^k X_j$  and  $M_1 = \overline{X_0}^X$ . Assume that we have defined  $M_j$  for  $j \leq i$ . Then we set

$$M_{i+1} = (\cup_{j=1}^k [\overline{(M_i - X_j)} \cap \overline{(M_i \cap X_j)}])^{-X}.$$

The same properties as in the single case hold. To be precise,  $\{M_i\}$  is a decreasing sequence of Nash set germs on  $X$ , for each  $i$  and  $j > 0$   $X_j \cap M_i - M_{i+1}$  is a union of some connected components of  $M_i - M_{i+1}$ , and  $\{M_i\}$  is canonical in the same sense as in the single case. We define also a *Nash germ decomposition* of  $\{X; X_j\}$  and a (the *canonical*) *global analytic germ decomposition* of a finite family of relatively compact global semianalytic sets in an analytic manifold in the same way. Then remark 3.3.(1) and 3.3.(2) hold also in the plural case.

Using these notions and remarks in the same way as above we can refine theorem 3.2 as follows.

**Remark 3.4.** In theorem 3.2, let  $A_i$  and  $B_i$  be a finite number of semialgebraic subsets of  $M$  such that the diffeomorphism germ of analytical right equivalence carries the germ on  $X$  of each  $A_i$  to the one of  $B_i$ . Then the diffeomorphism germ of Nash right equivalence is chosen to have the same property.

In particular, if we set  $f = g = \text{constant}$  then we have the following statement.

Let  $M$  and  $X$  be the same as in theorem 3.2, and let  $C_i$  and  $D_i$  be finitely many semialgebraic subsets of  $M$ . Assume that there exists a germ  $\pi$  on  $X$  of an analytic diffeomorphism between neighborhoods of  $X$  in  $M$  which carries the germ on  $X$  of each  $C_i$  to the one of  $D_i$  and such that  $\pi(X) = X$ . Then  $\pi$  is approximated by a germ on  $X$  of a Nash diffeomorphism between semialgebraic neighborhoods of  $X$  in  $M$  in the  $C^\infty$  topology keeping the properties of  $\pi$ .

### 3.2. Proof of theorem 1.11.

3.2.1. *Proof of theorem 1.11 in the case where  $X = M$  and  $M$  is compact.* Assume that  $f$  and  $g$  are almost Blow-analytically equivalent. Let  $\pi_f : N \rightarrow M$ ,  $\pi_g : L \rightarrow M$  and  $h : N \rightarrow L$  be two compositions of finite sequences of blowings-up along smooth analytic centers and an analytic diffeomorphism, respectively, such that  $f \circ \pi_f = g \circ \pi_g \circ h$ . We can assume that  $f \circ \pi_f$  and  $g \circ \pi_g$  have only normal crossing singularities, performing if necessary additional blowings-up.

Then by theorem 2.4 there exist compositions of finite sequences of blowings-up along smooth Nash centers  $\tilde{\pi}_f : \tilde{N} \rightarrow M$  and  $\tilde{\pi}_g : \tilde{L} \rightarrow M$  and analytic diffeomorphisms  $\tau_f : \tilde{N} \rightarrow N$  and  $\tau_g : \tilde{L} \rightarrow L$  such that  $f \circ \pi_f \circ \tau_f = f \circ \tilde{\pi}_f$  and  $g \circ \pi_g \circ \tau_g = g \circ \tilde{\pi}_g$ . Hence  $f \circ \tilde{\pi}_f \circ \tau_f^{-1} = g \circ \tilde{\pi}_g \circ \tau_g^{-1} \circ h$ , and  $f \circ \tilde{\pi}_f$  and  $g \circ \tilde{\pi}_g$  are analytically right equivalent. Then by theorem 3.2 they are Nash right equivalent, i.e.  $f$  and  $g$  are almost Blow-Nash equivalent.

Next we prove that if  $f$  and  $g$  are almost Blow-analytically R-L equivalent then they are almost Blow-Nash R-L equivalent. For that it suffices to prove that two analytically R-L equivalent Nash functions  $\phi$  and  $\psi$  with only normal crossing singularities are Nash R-L equivalent. Let  $\pi$  and  $\tau$  be analytic diffeomorphisms of  $M$  and  $\mathbf{R}$ , respectively, such that  $\tau \circ \phi = \psi \circ \pi$ . Then  $\pi(\text{Sing } \phi) = \text{Sing } \psi$ ,  $\tau(\phi(\text{Sing } \phi)) = \psi(\text{Sing } \psi)$  and  $\pi(\phi^{-1}(\phi(\text{Sing } \phi))) = \psi^{-1}(\psi(\text{Sing } \psi))$ . By remark 3.4 we have a Nash diffeomorphism  $\pi_0$  of  $M$  close to  $\pi$  in the  $C^\infty$  topology such that  $\pi_0(\text{Sing } \phi) = \text{Sing } \psi$  and  $\pi_0(\phi^{-1}(\phi(\text{Sing } \phi))) = \psi^{-1}(\psi(\text{Sing } \psi))$ , and since  $\phi(\text{Sing } \phi)$  is a finite set, we have a Nash diffeomorphism  $\tau_0$  of  $\mathbf{R}$  close to  $\tau$  in the



compact-open  $C^\infty$  topology such that  $\tau_0 = \tau$  on  $\phi(\text{Sing } \phi)$ . Replace  $\psi$  with  $\tau_0^{-1} \circ \psi \circ \pi_0$ . Then we can assume from the beginning that  $\text{Sing } \phi = \text{Sing } \psi$ ,  $\phi(\text{Sing } \phi) = \psi(\text{Sing } \psi)$ ,  $\phi^{-1}(\phi(\text{Sing } \phi)) = \psi^{-1}(\psi(\text{Sing } \psi))$ , and  $\pi$  and  $\tau$  are close to  $\text{id}$  in the  $C^\infty$  topology and in the compact-open  $C^\infty$  topology, respectively. Hence for each  $z_0 \in \phi(\text{Sing } \phi)$ ,  $\phi - z_0$  and  $\psi - z_0$  have the same sign at each point of  $M$  and the same multiplicity at each point of  $\phi^{-1}(z_0)$ . Let  $\rho$  be a Nash function on  $\mathbf{R}$  with zero set  $\phi(\text{Sing } \phi)$  and regular there. Then  $\rho \circ \phi$  and  $\rho \circ \psi$  satisfy the conditions in lemma 4.7, [6]— $(\rho \circ \phi)^{-1}(0) = (\rho \circ \psi)^{-1}(0) (= \phi^{-1}(\phi(\text{Sing } \phi)))$ ,  $\rho \circ \phi$  and  $\rho \circ \psi$  have the same sign at each point of  $M$ , only normal crossing singularities at  $(\rho \circ \phi)^{-1}(0)$  and the same multiplicity at each point of  $(\rho \circ \phi)^{-1}(0)$ , and the natural extension to  $M$  of the function  $\rho \circ \psi / \rho \circ \phi$  defined on  $M - (\rho \circ \phi)^{-1}(0)$  is close to 1 in the  $C^\infty$  topology. Hence by lemma 4.7 in [6] there exists a Nash diffeomorphism  $\pi_1$  of  $M$  close to  $\text{id}$  in the  $C^\infty$  topology such that  $\pi_1(\phi^{-1}(\phi(\text{Sing } \phi))) = \phi^{-1}(\phi(\text{Sing } \phi))$  and  $\phi - \psi \circ \pi_1$  is  $l$ -flat at  $\phi^{-1}(\phi(\text{Sing } \phi))$  for a large integer  $l$ . Replace, once more,  $\psi$  with  $\psi \circ \pi_1$ . Then we can assume, moreover, that  $\phi - \psi$  is  $l$ -flat at  $\phi^{-1}(\phi(\text{Sing } \phi))$  and close to 0 in the  $C^\infty$  topology. Hence by proposition 4.8,(i) in [6],  $\phi$  and  $\psi$  are analytically right equivalent and then by theorem 3.2 they are Nash right equivalent.

**3.2.2. Proof of theorem 1.11 in the case  $X \subset \text{Sing } f$ .** Assume that  $f$  and  $g$  are Nash functions defined on open semialgebraic neighborhoods  $U$  and  $V$ , respectively, of  $X$  in  $M$ , and let  $\pi_f : N \rightarrow U$ ,  $\pi_g : L \rightarrow V$  and  $h : N' \rightarrow L'$  be two compositions of finite sequences of blowings-up along smooth analytic centers and an analytic diffeomorphism from an open neighborhood  $N'$  of  $\pi_f^{-1}(X)$  in  $N$  to one  $L'$  of  $\pi_g^{-1}(X)$  in  $L$ , respectively, such that  $f \circ \pi_f = g \circ \pi_g \circ h$  and  $h(\pi_f^{-1}(X)) = \pi_g^{-1}(X)$ . When we proceed as in the proof in the case of  $X = M$  we can replace  $\pi_f : N \rightarrow U$ ,  $\pi_g : L \rightarrow V$  and  $h : N' \rightarrow L'$  by Nash  $\tilde{\pi}_f : \tilde{N} \rightarrow \tilde{U}$ ,  $\tilde{\pi}_g : \tilde{L} \rightarrow \tilde{V}$  and  $h : \tilde{N}' \rightarrow \tilde{L}'$ , respectively, so that  $f \circ \tilde{\pi}_f = g \circ \tilde{\pi}_g \circ \tilde{h}$ . However, we cannot expect the equality  $\tilde{h}(\tilde{\pi}_f^{-1}(X)) = \tilde{\pi}_g^{-1}(X)$ . For the equality we need to modify  $\pi_f$  and  $\pi_g$ .

As in the construction of the canonical Nash germ decomposition we have a decreasing sequence of Nash sets  $X_i$ ,  $i = 1, 2, \dots$ , in  $U$  such that  $X_1$  is the Nash closure of  $X$  in  $U$  and for each  $i$  the set  $X_i \cap X - X_{i+1}$  is a union of some connected components of  $X_i - X_{i+1}$ . Set  $X_{f,i} = \pi_f^{-1}(X_i)$ . Then  $\{X_{f,i}\}$  is a decreasing sequence of global analytic sets in  $N$ ,  $\pi_f^{-1}(X) \subset X_{f,1}$ , and for each  $i$  the set  $\pi_f^{-1}(X) \cap X_{f,i} - X_{f,i+1}$  is a union of some connected components of  $X_{f,i} - X_{f,i+1}$ .

Now, by Hironaka Desingularization Theorem (and similarly to the case of  $X = M$ ), we are able to reduce the problem to the case where  $X_{f,i}$  are normal crossing,  $f \circ \pi_f$  has only normal crossing singularities, and hence  $\pi_f^{-1}(X)$  is a union of some connected components of strata of the canonical stratification of  $\text{Sing}(f \circ \pi_f)$ . We call these properties (\*). Shrinking  $N'$ ,  $V$ ,  $L$  and  $L'$  so that  $L' = L$  if necessary, then  $\pi_g^{-1}(X)$  and  $g \circ \pi_g$  satisfy also (\*).

Let  $N$  and  $L$  be realized in  $U \times \mathbf{P}(k_{f,0}) \times \dots \times \mathbf{P}(k_{f,m'-1})$  and in  $V \times \mathbf{P}(k_{g,0}) \times \dots \times \mathbf{P}(k_{g,m''-1})$ , respectively, as in theorem 2.4. Then by theorem 2.4 there exist compositions of finite sequences of blowings-up along smooth Nash centers  $\tilde{\pi}_f : \tilde{N} \rightarrow \tilde{U}$  and  $\tilde{\pi}_g : \tilde{L} \rightarrow \tilde{V}$  and analytic embeddings  $h_f : \tilde{N} \rightarrow N'$  and  $h_g : \tilde{L} \rightarrow L'$  such that

- $\tilde{U}$  and  $\tilde{V}$  are open semialgebraic neighborhoods of  $X$  in  $U$  and  $V$ , respectively,
- $f \circ \pi_f \circ h_f = f \circ \tilde{\pi}_f$ ,  $g \circ \pi_g \circ h_g = g \circ \tilde{\pi}_g$ ,  $\tilde{N}$  and  $\tilde{L}$  are realized in  $\tilde{U} \times \mathbf{P}(k_{f,0}) \times \dots \times \mathbf{P}(k_{f,m'-1})$  and in  $\tilde{V} \times \mathbf{P}(k_{g,0}) \times \dots \times \mathbf{P}(k_{g,m''-1})$ , respectively,
- $\tilde{N}$  and  $\tilde{L}$  are close to  $N$  and  $L$  at  $\tilde{\pi}_f^{-1}(X)$  and  $\tilde{\pi}_g^{-1}(X)$ , respectively, in the  $C^\infty$  topology,
- (\*\*)  $h_f$  and  $h_g$  are close to  $\text{id}$  at  $\tilde{\pi}_f^{-1}(X)$  and  $\tilde{\pi}_g^{-1}(X)$ , respectively, in the  $C^\infty$  topology.

Hence

$$f \circ \tilde{\pi}_f \circ h_f^{-1} = g \circ \tilde{\pi}_g \circ h_g^{-1} \circ h \quad \text{on } \text{Im } h_f \cap h^{-1}(\text{Im } h_g).$$

Clearly  $h_f(\text{Sing}(f \circ \tilde{\pi}_f)) \subset \text{Sing}(f \circ \pi_f)$  and  $h_g(\text{Sing}(g \circ \tilde{\pi}_g)) \subset \text{Sing}(g \circ \pi_g)$ . It follows from (\*) and (\*\*) that  $\tilde{\pi}_f^{-1}(X)$  and  $\tilde{\pi}_g^{-1}(X)$  are unions of some connected components of strata of the canonical stratifications of  $\text{Sing}(f \circ \tilde{\pi}_f)$  and  $\text{Sing}(g \circ \tilde{\pi}_g)$ , respectively, and hence

$$h_f(\tilde{\pi}_f^{-1}(X)) = \pi_f^{-1}(X) \quad \text{and} \quad h_g(\tilde{\pi}_g^{-1}(X)) = \pi_g^{-1}(X).$$

Therefore, the germs of  $f \circ \tilde{\pi}_f$  on  $\tilde{\pi}_f^{-1}(X)$  and of  $g \circ \tilde{\pi}_g$  on  $\tilde{\pi}_g^{-1}(X)$  are analytically right equivalent. On the other hand, by remark 3.4, the germs of  $\tilde{N}$  on  $\tilde{\pi}_f^{-1}(X)$  and of  $\tilde{L}$  on  $\tilde{\pi}_g^{-1}(X)$  are Nash diffeomorphic. Hence we can regard them as the same Nash set germ. Then by theorem 3.2 and remark 3.4, the germs of  $f \circ \tilde{\pi}_f$  on  $\tilde{\pi}_f^{-1}(X)$  and of  $g \circ \tilde{\pi}_g$  on  $\tilde{\pi}_g^{-1}(X)$  are Nash right equivalent. Thus the germs of  $f$  and  $g$  on  $X$  are almost Blow-Nash equivalent.

Finally, the case of the R-L equivalences runs in the same way as that of  $X = M$ .

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